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Generalized fractional minimax programming with B-(p, r)-invexity

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ABSTRACT

Optimality conditions are proved for a class of generalized fractional minimax programming problems involving B-(p, r)-invexity functions. Subsequently, these optimality conditions are utilized as a basis for constructing various duality models for this type of fractional programming problems and proving appropriate duality theorems.

1. Introduction

Optimization problems, in which both a minimization and a maximization process of fractional objectives are performed, are usually referred in the optimization literature as generalized fractional minimax programming problems. These problems have arisen in multiobjective programming [1,2], game theory [3], goal programming [4], minimum risk problems [5], and economics [6,7]. Problems of these type have been the subject of immense interest in the past few years. Recently, optimality conditions and various duality results have been obtained for minimax fractional programming problems involving the optimization several ratios in the objective function. The necessary and sufficient conditions for generalized minimax programming were first developed by Schmitendorf [8]. Later Tanimoto [9] proved duality theorems, under a convexity assumption on the functions involved, for the problems considered by Schmitendorf. Crouzeix et al. [10] have given a variety of applications of generalized fractional programming and have shown that the minimax fractional program can be solved by solving a minimax parametric program, while Jagannathan and Schaible [11] obtained various duality results for such a problem via Farkas' lemma. Bector et al. [12] have developed duality for the generalized minimax fractional program, under generalized convexity assumption, using a minimax parametric program. More recently, a number of optimality criteria, duality relations, and computational algorithms for various classes of generalized fractional programming problems have appeared in the related literature (for a bibliography of fractional programming, see [13]).

The above-mentioned authors have developed optimality conditions and duality theory for fractional problems under convexity conditions and later under various generalized convexity assumptions and using different approaches (parametrization, Farkas Lemma, Mond-Weir duality, Wolfe duality, F.John or Karush–Kuhn–Tucker optimality conditions). In the recent years, quite a number of publications appeared, in which various duality theorems are proved under weaker assumptions than convexity imposed on the functions constituting multiobjective fractional programming problems. Zalmai [14] proved several sufficient optimality conditions and various duality theorems for a class of minimax programming problems in Banach space under the generalized invexity assumption. In [15], Zalmai established both parametric and nonparametric necessary and sufficient optimality conditions for a class of nonsmooth generalized fractional programming problems containing the so-called ρ-convex functions. Subsequently, he employed these optimality conditions to construct both parametric and nonparametric duality models and he established appropriate duality theorems. Liu [16] extended the

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results of Bector et al. [12] to nonsmooth generalized fractional programming problems containing V-pseudoinvex functions. In [17], Antczak established both parametric and non-parametric sufficient optimality conditions and construct several parametric and parameter-free duality models for the generalized minimax fractional program, under the (p, r)-invexity assumption.

In the paper, we shall establish both parametric and non-parametric sufficient optimality conditions and construct several parametric and parameter-free duality models for the generalized fractional minimax program. The dual models have been motivated by those due to Mond and Weir [18] for scalar fractional programming and for generalized fractional programming (see, for example, Bector et al. [12], Singh and Rueda [19]). For obtaining sufficient conditions and various duality results, we will make use of the assumption that all functions constituting a generalized fractional minimax programming problem and its dual problems are B-(p, r)-invex with respect to the same function η and with respect to, not necessarily, the same function b. Based on this property, for each dual model considered in the paper, its modified form is also presented with respect to the function η and the scalar p (dual models of this type are denoted in the paper by index p). In this way, we present no one dual model of a specified type for the considered generalized fractional minimax program, but we define various classes of dual models of the same specified type (with respect to the function η and the scalar p). In particular, these classes of dual models, do not seem to have been considered previously for any type of generalized fractional minimax problems.

2. Preliminaries

Hanson [20] defined invex functions which allow the use of the Karush–Kuhn–Tucker conditions as sufficient conditions for optimality in constrained optimization problems. Moreover, weak duality in the sense of Wolfe holds between the primal problem and its associated Wolfe dual problem under the invexity assumption. Later, Craven [21] named those functions, as invex functions.

Hanson's initial results inspired a great deal of subsequent work which has greatly expanded the role of invexity in optimization. Generalizing invex functions, Antczak [22] defined several classes of (p, r)-invex functions. The following definition generalizes the definition of a class of (p, r)-invex functions [22] to the case of a class of B-(p, r)-invex functions.

Definition 1 (*[23]*). The differentiable function $f : X \to R$ is said to be (strictly) B-(p, r)-invex with respect to η and b at $u \in X$ on a nonempty set $X \subset R^n$ if, there exist a function $\eta : X \times X \to R^n$ and a function $b : X \times X \to R_+$ such that, for all $x \in X$, the inequality

$$\begin{aligned}
&\frac{1}{r}b(x,u)\left(e^{r(f(x)-f(u))}-1\right) \geq \frac{1}{p}\nabla f(u)\left(e^{p\eta(x,u)}-\mathbf{1}\right) && (> \text{ if } x \neq u) \text{ for } p \neq 0, r \neq 0,\\
&\frac{1}{r}b(x,u)\left(e^{r(f(x)-f(u))}-1\right) \geq \nabla f(u)\eta(x,u) && (> \text{ if } x \neq u) \text{ for } p = 0, r \neq 0,\\
&b(x,u)\left(f(x)-f(u)\right) \geq \frac{1}{p}\nabla f(u)\left(e^{p\eta(x,u)}-\mathbf{1}\right) && (> \text{ if } x \neq u) \text{ for } p \neq 0, r = 0,\\
&b(x,u)\left(f(x)-f(u)\right) \geq \nabla f(u)\eta(x,u) && (> \text{ if } x \neq u) \text{ for } p = 0, r = 0,
\end{aligned}$$

holds.

f is said to be (strictly) B-(p, r)-invex with respect to η and b on X if it is B-(p, r)-invex with respect to the same η and b at each $u \in X$ on X.

Remark 2. It should be pointed out that the exponentials appearing on the right-hand sides of the inequalities above are understood to be taken componentwise and $\mathbf{1} = (1, 1, \ldots, 1) \in R^n$.

Remark 3. For some properties of a class of B-(p, r)-invex functions, the readers are advised to consult [23].

Remark 4. In order to define an analogous class of (strictly) B-(p, r)-incave functions with respect to η and b, the direction of the inequality in the definition of these functions should be reversed.

Remark 5. All theorems in the rest of this work will be proved only in the case when $p \neq 0$, $r \neq 0$ (other cases be dealt with similarly since the only changes arise from the form of inequality defining the class of B-(p, r)-invex functions with respect to η for given p and r). The proofs in the other cases are easier than in this one. It follows from the form of inequalities which are given in Definition 1. Moreover, without limiting the generality considerations we shall assume that $r > 0$ (in the case when $r < 0$ the direction some of the inequalities in the proofs of theorems should be changed to the opposite one).

3. Generalized fractional minimax programming

In this section, we shall establish parametric and nonparametric sufficient optimality conditions for a class of smooth generalized fractional programming problems containing B-(p, r)-invex functions.

In this paper, we consider the following generalized fractional minimax programming problem:

$$\begin{aligned} &\text{Minimize } \Phi(x) = \sup_{y \in Y} \frac{f(x,y)}{g(x,y)} \\ &\text{subject to } h_j(x) \leq 0, \quad j = 1, \ldots, k, \end{aligned} \tag{FP}$$

where X is a nonempty subset of R^n, Y is a specified subset of R^m, $f : X \times Y \to R$, $g : X \times Y \to R$ are C^1 on $X \times Y$, $h : X \to R$ is C^1 on X.

Let $D := \{x \in X : h_j(x) \leq 0, \ j \in J = 1, \ldots, k\}$ (assumed to be nonempty) denotes the feasible set of (FP). Further, we assume that $f(x,y) \geq 0$ and $g(x,y) > 0$ for all $(x,y) \in D \times Y$.

Throughout this paper we also assume that Y is a compact set. This means that for every $\widetilde{x} \in D$ there exists $\widetilde{y} \in Y$ with the following property: $\frac{f(\widetilde{x},\widetilde{y})}{g(\widetilde{x},\widetilde{y})} = \sup_{y \in Y} \frac{f(\widetilde{x},y)}{g(\widetilde{x},y)}$.

We denote by $J(x)$ the set of active constraints at $x \in D$, that is,

$$J(x) = \{j \in J : h_j(x) = 0\}.$$

Further, let

$$Y(x) = \left\{ y \in Y : \frac{f(x,y)}{g(x,y)} = \sup_{q \in Y} \frac{f(x,q)}{g(x,q)} \right\}.$$

In the sequel, we shall use the following theorem proved by Chandra and Kumar [24]:

Theorem 6 (*Necessary Optimality Conditions*). *Let $\overline{x}$ be an optimal solution in* (FP) *and $\nabla h_j(\overline{x})$, $j \in J(\overline{x})$ be linearly independent* [25]. *Then there exist a positive integer $\overline{\alpha}$ such that $1 \leq \overline{\alpha} \leq n+1$, scalars $\overline{\lambda}_i$, $i = 1, \ldots, \overline{\alpha}$, scalars $\overline{\xi}_j$, $j = 1, \ldots, k$, vectors $\overline{y}^i$, $i = 1, \ldots, \overline{\alpha}$, a scalar $\overline{v}$, such that*

$$\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \left(\nabla f\left(\overline{x}, \overline{y}^i\right) - \overline{v} \nabla g\left(\overline{x}, \overline{y}^i\right) \right) + \sum_{j=1}^{k} \overline{\xi}_j \nabla h_j(\overline{x}) = 0, \tag{1}$$

$$f\left(\overline{x}, \overline{y}^i\right) - \overline{v} g\left(\overline{x}, \overline{y}^i\right) = 0, \quad i = 1, \ldots, \overline{\alpha}, \tag{2}$$

$$\sum_{j=1}^{k} \overline{\xi}_j h_j(\overline{x}) = 0, \tag{3}$$

$$\overline{\lambda}_i \geq 0, \quad \sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i = 1, \qquad \overline{y}^i \in Y(\overline{x}), \quad i = 1, \ldots, \overline{\alpha}, \qquad \overline{\xi}_j \geq 0, \quad j = 1, \ldots, k. \tag{4}$$

Now, we prove the following sufficient optimality conditions under the assumption of B-(p,r)-invexity imposed on the functions constituting the considered generalized minimax fractional optimization problem (FP).

Theorem 7. *Let $\overline{x}$ be a feasible solution in* (FP). *We assume that there exist an integer number $\overline{\alpha}$, $1 \leq \overline{\alpha} \leq n+1$, scalars $\overline{\lambda}_i$, $i = 1, \ldots, \overline{\alpha}$, scalars $\overline{\xi}_j$, $j = 1, \ldots, k$, vectors $\overline{y}^i \in Y(\overline{x})$, $i = 1, \ldots, \overline{\alpha}$, such that the conditions* (1)–(4) *are fulfilled, $\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \left(f\left(\cdot, \overline{y}^i\right) - \overline{v} g\left(\cdot, \overline{y}^i\right)\right)$ is B-(p,r)-invex at $\overline{x}$ on D with respect to η and b satisfying $b(x,\overline{x}) > 0$ for all $x \in D$, and, moreover, $\sum_{j=1}^{k} \overline{\xi}_j h_j(\cdot)$ is B_h-(p,r)-invex at $\overline{x}$ on D with respect to the same function η, and with respect to the function b_h, not necessarily, equal to b. Then $\overline{x}$ is an optimal solution in* (FP).

Proof. Let $\overline{x}$ be an arbitrary feasible solution in (FP). Moreover, we assume that there exist an integer number $\overline{\alpha}$, $1 \leq \overline{\alpha} \leq n+1$, scalars $\overline{\lambda}_i$, vectors $\overline{y}^i \in Y(\overline{x})$, $i = 1, \ldots, \overline{\alpha}$, scalars $\overline{\xi}_j$, $j = 1, \ldots, k$, such that the conditions (1)–(4) are fulfilled at $\overline{x}$. We proceed by contradiction. Suppose that $\overline{x}$ is not optimal in (FP). Then there exists a feasible solution $\widetilde{x}$ in (FP), such that

$$\overline{v} = \sup_{y \in Y} \frac{f(\overline{x},y)}{g(\overline{x},y)} > \sup_{y \in Y} \frac{f(\widetilde{x},y)}{g(\widetilde{x},y)}. \tag{5}$$

Thus,

$$\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \left(f\left(\widetilde{x}, \overline{y}^i\right) - \overline{v} g\left(\widetilde{x}, \overline{y}^i\right) \right) < 0. \tag{6}$$

Using (2) together with (4) and (6), we obtain

$$\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \left(f\left(\widetilde{x}, \overline{y}^i\right) - \overline{v} g\left(\widetilde{x}, \overline{y}^i\right) \right) < \sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \left(f\left(\overline{x}, \overline{y}^i\right) - \overline{v} g\left(\overline{x}, \overline{y}^i\right) \right). \tag{7}$$

By assumption, $\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \left(f\left(\cdot, \overline{y}^i\right) - \overline{v}g\left(\cdot, \overline{y}^i\right)\right)$ is B-(p, r)-invex at $\overline{x}$ on D with respect to η and b. Then, by Definition 1, the following inequality

$$\frac{1}{r}b(x, \overline{x}) \left(e^{r\left[\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i\left(f(x,\overline{y}^i)-\overline{v}g(x,\overline{y}^i)\right)-\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i\left(f(\overline{x},\overline{y}^i)-\overline{v}g(\overline{x},\overline{y}^i)\right)\right]} - 1\right) \geq \frac{1}{p}\left(\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \left(\nabla f(\overline{x}, \overline{y}^i) - \overline{v}\nabla g(\overline{x}, \overline{y}^i)\right)\right)\left(e^{p\eta(x,\overline{x})} - \mathbf{1}\right)$$

holds for all $x \in D$, and so for $x = \widetilde{x}$. Using $b(\widetilde{x}, \overline{x}) > 0$ together with the inequality above, we get

$$\frac{1}{p}\left(\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \left(\nabla f_i(\overline{x}, \overline{y}^i) - \overline{v}\nabla g_i(\overline{x}, \overline{y}^i)\right)\right)\left(e^{p\eta(\widetilde{x},\overline{x})} - \mathbf{1}\right) < 0. \tag{8}$$

From the feasibility of $\widetilde{x}$ together with $\overline{\xi}_j \geq 0, j \in J$, we have

$$\sum_{j=1}^{k} \overline{\xi}_j h_j\left(\widetilde{x}\right) \leq 0. \tag{9}$$

By assumption, $\sum_{j=1}^{k} \overline{\xi}_j h_j(\cdot)$ is B_h-(p, r)-invex at $\overline{x}$ on D with respect to η and with respect to b_h. Since $b_h(x, \overline{x}) \geq 0$ for all $x \in D$ then by (3), we obtain

$$\frac{1}{r}b_h\left(\widetilde{x}, \overline{x}\right)\left(e^{r\left(\sum_{j=1}^{k} \overline{\xi}_j h_j(\widetilde{x})-\sum_{j=1}^{k} \overline{\xi}_j h_j(\overline{x})\right)} - 1\right) \leq 0.$$

Hence, by Definition 1, the above inequality implies

$$\frac{1}{p}\sum_{j=1}^{k} \overline{\xi}_j \nabla h_j(\overline{x})\left(e^{p\eta(\widetilde{x},\overline{x})} - \mathbf{1}\right) \leq 0. \tag{10}$$

Thus, by (8) and (10), we obtain the inequality

$$\frac{1}{p}\left(\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \left(\nabla f(\overline{x}, \overline{y}^i) - \overline{v}\nabla g(\overline{x}, \overline{y}^i)\right) + \sum_{j=1}^{k} \overline{\xi}_j \nabla h_j(\overline{x})\right)\left(e^{p\eta(\widetilde{x},\overline{x})} - \mathbf{1}\right) < 0, \tag{11}$$

which contradicts (1). ■

We define the so-called α-reduced Lagrange function for problem (FP) in the form

$$L_\alpha\left(x, y, \lambda, \xi, v\right) := \sum_{i=1}^{\alpha} \lambda_i \left(f(x, y^i) - vg(x, y^i)\right) + \sum_{j=1}^{k} \xi_j h_j(x), \tag{12}$$

where α ranges over the integers such that $1 \leq \alpha \leq n + 1$.

It turns out that the sufficient conditions can be proved under the assumption of B-(p, r)-invexity imposed on the α-reduced Lagrange function defined above.

Theorem 8 (*Sufficient Optimality Conditions*). *Let $\overline{x}$ be a feasible solution in* (FP) *and the necessary optimality conditions* (1)–(4) *be fulfilled at $\overline{x}$. Moreover, assume that the $\overline{\alpha}$-reduced Lagrangian in* (FP) *is B-(p, r)-invex at $\overline{x}$ on D with respect to η and b satisfying the following condition: $b\left(x, \overline{x}\right) > 0$ for all $x \in D$. Then $\overline{x}$ is an optimal solution in* (FP).

Proof. Let $\overline{x}$ be an arbitrary feasible solution in (FP). Moreover, there exist an integer number $\overline{\alpha}$, $1 \leq \overline{\alpha} \leq n + 1$, scalars $\overline{\lambda}_i$, y^i, $i = 1, \ldots, \overline{\alpha}$, scalars $\overline{\xi}_j$, $j = 1, \ldots, k$, such that the conditions (1)–(4) are fulfilled at $\overline{x}$. Since the $\overline{\alpha}$-reduced Lagrangian of (FP) is B-(p, r)-invex at $\overline{x}$ on D with respect to η and b then, by Definition 1, the following inequality

$$\frac{1}{r}b(x, \overline{x})\left(e^{r\left(L_{\overline{\alpha}}\left(x,\overline{y},\overline{\lambda},\overline{\xi},\overline{v}\right)-L_{\overline{\alpha}}\left(\overline{x},\overline{y},\overline{\lambda},\overline{\xi},\overline{v}\right)\right)} - 1\right) \geq \frac{1}{p}\nabla L_{\overline{\alpha}}\left(\overline{x}, \overline{y}, \overline{\lambda}, \overline{\xi}, \overline{v}\right)\left(e^{p\eta(x,\overline{x})} - \mathbf{1}\right) \tag{13}$$

holds for all $x \in D$. Thus, by the Karush–Kuhn–Tucker necessary optimality condition (1) and since $b(x, \overline{x}) > 0$ for all $x \in D$, the following inequality

$$L_{\overline{\alpha}}\left(x, \overline{y}, \overline{\lambda}, \overline{\xi}, \overline{v}\right) \geq L_{\overline{\alpha}}\left(\overline{x}, \overline{y}, \overline{\lambda}, \overline{\xi}, \overline{v}\right) \tag{14}$$

holds for all $x \in D$. Using the Karush–Kuhn–Tucker necessary optimality conditions (2)–(3), from the definition of the α-reduced Lagrange function it follows that $L_{\overline{\alpha}}\left(\overline{x}, \overline{y}, \overline{\lambda}, \overline{\xi}, \overline{v}\right) = 0$. Hence, (14) gives

$$\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \left(f(x, \overline{y}^i) - \overline{v}g(x, \overline{y}^i)\right) + \sum_{j=1}^{k} \overline{\xi}_j h_j(x) \geq 0.$$

From $x \in D$ and by the Karush–Kuhn–Tucker necessary optimality condition (4), it follows that the following inequality

$$\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \left(f(x, \overline{y}^i) - \overline{v} g(x, \overline{y}^i) \right) \geq 0$$

holds for all $x \in D$. Since $\overline{\lambda}_i \geq 0$ for $i = 1, \ldots, \overline{\alpha}$, therefore, there exists i^* such that

$$f(x, \overline{y}^{i^*}) - \overline{v} g(x, \overline{y}^{i^*}) \geq 0$$

for all $x \in D$. Hence,

$$\sup_{y \in Y} \frac{f(x, y)}{g(x, y)} \geq \overline{v},$$

so that,

$$\sup_{y \in Y} \frac{f(x, y)}{g(x, y)} \geq \sup_{y \in Y} \frac{f(\overline{x}, y)}{g(\overline{x}, y)}.$$

This means that $\overline{x}$ is optimal in problem (FP). ■

Remark 9. Analyzing proofs of Theorems 7 and 8, in fact, we can prove the sufficient optimality conditions under weaker assumptions than conditions (1)–(4). Namely, it is sufficient to assume, in place of (1), the following condition:

$$\begin{cases} \frac{1}{p} \left(\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \left(\nabla f_i(\overline{x}, y^i) - \overline{v} \nabla g(\overline{x}, y^i) \right) + \sum_{j=1}^{m} \overline{\xi}_j \nabla h_j(\overline{x}) \right) \left(e^{p\eta(x,\overline{x})} - \mathbf{1} \right) \geq 0 & \text{if } p \neq 0 \\ \left(\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \left(\nabla f_i(\overline{x}, y^i) - \overline{v} \nabla g(\overline{x}, y^i) \right) + \sum_{j=1}^{m} \overline{\xi}_j \nabla h_j(\overline{x}) \right) \eta(x, \overline{x}) \geq 0 & \text{if } p = 0 \end{cases} \quad \text{for all } x \in D.$$

In order to discuss various nonparametric dual models for the considered generalized fractional minimax programming problem (FP), we state another version of Theorem 6. This can be accomplished by simply replacing the parameter $\overline{v}$ with $f(\overline{x}, \overline{y}^i)/g(\overline{x}, \overline{y}^i)$ and rewriting the multiplier functions associated with the inequality constraints (see, for example, [12,15]).

Theorem 10 (*Nonparametric Necessary Optimality Conditions*). *Let $\overline{x}$ be an optimal solution in* (FP) *and $\nabla h_j(\overline{x})$, $j \in J(\overline{x})$ be linearly independent* [25]. *Then there exist a positive integer $\overline{\alpha}$, scalars $\overline{\lambda}_i \geq 0$, $i = 1, \ldots, \overline{\alpha}$, scalars $\overline{\xi}_j \geq 0$, $j = 1, \ldots, k$, vectors $\overline{y}^i$, $i = 1, \ldots, \overline{\alpha}$, such that*

$$\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \left(\nabla f\left(\overline{x}, \overline{y}^i\right) g\left(\overline{x}, \overline{y}^i\right) - f\left(\overline{x}, \overline{y}^i\right) \nabla g\left(\overline{x}, \overline{y}^i\right) \right) + \sum_{j=1}^{k} \overline{\xi}_j \nabla h_j(\overline{x}) = 0, \tag{15}$$

$$\sum_{j=1}^{k} \overline{\xi}_j h_j(\overline{x}) = 0, \tag{16}$$

$$\overline{\lambda}_i \geq 0, \quad \sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i = 1, \quad \overline{y}^i \in Y(\overline{x}), \quad i = 1, \ldots, \overline{\alpha}, \quad \overline{\xi}_j \geq 0, \quad j = 1, \ldots, k. \tag{17}$$

4. Duality

In this section, we present various duality models. To prove duality results between the primal problem (FP) and each constructed dual models, we assume that the functions constituting these problems are B-(p, r)-invex with respect to the same function η and with respect to, not necessarily, the same function b.

In this section, let A denote the set of triples (α, λ, y), where α ranges over the integers such that $1 \leq \alpha \leq n + 1$, $\lambda \in R_+^{\alpha}$, $\sum_{i=1}^{\alpha} \lambda_i = 1$, and $\overline{y} = (y^1, \ldots, y^{\alpha})$ is an $m\alpha$-dimensional vector with $y^i \in Y(x)$ for all $i = 1, \ldots, \alpha$ and for some $x \in R^n$.

4.1. Schaible type dual

Now, we formulate the duals in the format of Schaible and prove appropriate duality theorems between the primal problem (FP) and the constructed Schaible type dual models.

We define $H_1(\alpha, \lambda, \bar{y})$ as the set of all triples $(u, \xi, v) \in X \times R_+^k \times R_+$ satisfying the following conditions:

$$\sum_{i=1}^{\alpha} \lambda_i \left(\nabla f\left(u, y^i\right) - v\nabla g\left(u, y^i\right)\right) + \sum_{j=1}^{k} \xi_j \nabla h_j(u) = 0, \tag{18}$$

$$f\left(u, y^i\right) - vg\left(u, y^i\right) \geq 0, \quad i = 1, \ldots, \alpha, \tag{19}$$

$$\sum_{j=1}^{k} \xi_j h_j(u) \geq 0, \tag{20}$$

$$\lambda_i \geq 0, \quad \sum_{i=1}^{\alpha} \lambda_i = 1, \qquad y^i \in Y(u), \quad i = 1, \ldots, \alpha, \qquad \xi_j \geq 0, \quad j = 1, \ldots, k. \tag{21}$$

Now, following Schaible, we consider a dual problem (FD1) to the generalized fractional minimax problem (FP) as follows:

$$\max_{(\alpha,\lambda,\bar{y})\in A} \ \sup_{(u,\xi,v)\in H_1(\alpha,\lambda,\bar{y})} v$$

$$\nabla\left[\sum_{i=1}^{\alpha} \lambda_i \left(f_i(u) - vg_i(u)\right) + \sum_{j=1}^{k} \xi_j h_j(u)\right] = 0, \qquad \text{(FD1)} \tag{22}$$

$$\sum_{i=1}^{\alpha} \lambda_i \left(f(u, y^i) - vg(u, y^i)\right) + \sum_{j=1}^{k} \xi_j h_j(u) \geq 0, \tag{23}$$

$$\lambda_i \geq 0, \quad \sum_{i=1}^{\alpha} \lambda_i = 1, \qquad y^i \in Y(u), \quad i = 1, \ldots, \alpha, \qquad \xi_j \geq 0, \quad j = 1, \ldots, k. \tag{24}$$

The Schaible dual problem we also define as follows:

$$\max_{(\alpha,\lambda,\bar{y})\in A} \ \sup_{(u,\xi,v)\in H_1(\alpha,\lambda,\bar{y})} v$$

subject to (23)– (24) and (25) in place of (22) (FD1p)

$$\left.\begin{array}{l} \dfrac{1}{p}\nabla\left[\displaystyle\sum_{i=1}^{\alpha} \lambda_i \left(f(u, y^i) - vg(u, y^i)\right) + \sum_{j=1}^{k} \xi_j h_j(u)\right]\left(e^{p\eta(x,u)} - \mathbf{1}\right) \geq 0 \quad \text{if } p \neq 0 \\ \nabla\left[\displaystyle\sum_{i=1}^{\alpha} \lambda_i \left(f(u, y^i) - vg(u, y^i)\right) + \sum_{j=1}^{k} \xi_j h_j(u)\right]\eta(x, u) \geq 0 \quad \text{if } p = 0 \end{array}\right\} \text{for all } x \in D. \tag{25}$$

Thus, by (23)–(25), we define a class of Schaible dual problems with respect to the function η and the scalar p.
If, for a triplet $(\alpha, \lambda, \bar{y}) \in A$, the set $H_1(\alpha, \lambda, \bar{y})$ is empty, then we define the supremum over it to be $-\infty$.
Let W_{FD1} denote a set of all feasible solutions for problem (FD1). Moreover, let U_1 denote

$$U_1 = \{u \in X : (\alpha, \lambda, \bar{y}, u, \xi, v) \in W_{FD1}\}.$$

Theorem 11 (*Weak Duality*). *Let x and $(\alpha, \lambda, y, u, \xi, v)$ be feasible solutions for* (FP) *and* (FD1), *respectively. If $\sum_{i=1}^{\alpha} \lambda_i(f(\cdot, y^i) - vg(\cdot, y^i))$ is B-(p, r)-invex at u on $D \cup U_1$ with respect to η and b satisfying $b(x, u) > 0$ and, moreover, $\sum_{j=1}^{k} \xi_j h_j(\cdot)$ is B_h-(p, r)-invex at u on $D \cup U_1$ with respect to the same function η and with respect to the function b_h, not necessarily, equal to b. Then*

$$\sup_{y\in Y} \frac{f(x, y)}{g(x, y)} \geq v.$$

Proof. Using the feasibility of x in (FP) together with (24), we have

$$\sum_{j=1}^{k} \xi_j h_j(x) \leq 0. \tag{26}$$

By assumption, $\sum_{j=1}^{k} \xi_j h_j(\cdot)$ is B_h-(p, r)-invex at u on $D \cup U_1$ with respect to η and with respect to b_h. Then, by Definition 1, there exists a function b_h such that $b_h(x, u) \geq 0$. By (20) and (26), it follows that

$$\frac{1}{r} b_h(x, u) \left(e^{r\left(\sum_{j=1}^{k} \xi_j h_j(x) - \sum_{j=1}^{k} \xi_j h_j(u)\right)} - 1\right) \leq 0.$$

Then by Definition 1, we get

$$\frac{1}{p} \sum_{j=1}^{k} \xi_j \nabla h_j(u) \left(e^{p\eta(x,u)} - \mathbf{1}\right) \leq 0. \tag{27}$$

Hence, by (18), we obtain the inequality

$$\frac{1}{p}\sum_{i=1}^{\alpha}\lambda_i\left(\nabla f(u,y^i)-v\nabla g(u,y^i)\right)\left(e^{p\eta(x,u)}-\mathbf{1}\right)\geq 0. \tag{28}$$

By assumption, $\sum_{i=1}^{\alpha}\lambda_i\left(f(\cdot,y^i)-vg(\cdot,y^i)\right)$ is B-(p,r)-invex with respect to η and b at u on $D\cup U_1$. Then, by Definition 1, it follows that

$$\frac{1}{r}b(x,u)\left(e^{r\left(\sum_{i=1}^{\alpha}\lambda_i\left[f(x,y^i)-vg(x,y^i)\right]-\sum_{i=1}^{\alpha}\lambda_i\left[f(u,y^i)-vg(u,y^i)\right]\right)}-1\right)\geq 0,$$

so that,

$$\sum_{i=1}^{\alpha}\lambda_i\left[f(x,y^i)-vg(x,y^i)\right]\geq\sum_{i=1}^{\alpha}\lambda_i\left[f(u,y^i)-vg(u,y^i)\right]. \tag{29}$$

Thus, by (19),

$$\sum_{i=1}^{\alpha}\lambda_i\left[f(x,y^i)-vg(x,y^i)\right]\geq 0.$$

Since $\lambda_i\geq 0$ for $i=1,\ldots,\alpha$, therefore, there exists i^* such that

$$f(x,y^{i^*})-vg(x,y^{i^*})\geq 0.$$

Hence,

$$\sup_{y\in Y}\frac{f(x,y)}{g(x,y)}\geq\frac{f(x,y^{i^*})}{g(x,y^{i^*})}\geq v.$$

This completes the proof. ■

Theorem 12 (*Strong Duality*). *Let $\bar{x}$ be an optimal point in* (FP) *and the Linear Independence Constraint Qualification (LICQ)* [25] *be satisfied at $\bar{x}$. Then, there exist $\left(\bar{\alpha},\bar{\lambda},\bar{y}\right)\in A$ and $(\bar{x},\bar{\xi},\bar{v})\in H_1\left(\bar{\alpha},\bar{\lambda},\bar{y}\right)$ such that $\left(\bar{\alpha},\bar{\lambda},\bar{y},\bar{x},\bar{\xi},\bar{v}\right)$ is optimal in* (FD1). *If also the hypotheses of Theorem 11 hold, then the corresponding optimal values of* (FP) *and* (FD1) *are equal.*

Proof. By assumption, $\bar{x}$ is an optimal point of (FP) and the Linear Independence Constraint Qualification (LICQ) is satisfied at $\bar{x}$. Then, by Theorem 6, there exist a positive integer $\bar{\alpha}$, scalars $\bar{\lambda}_i\geq 0$, $i=1,\ldots,\bar{\alpha}$, scalars $\bar{\xi}_j\geq 0$, $j=1,\ldots,k$, vectors $\bar{y}^i\in Y(\bar{x})$, $i=1,\ldots,\bar{\alpha}$, such that $\left(\bar{\alpha},\bar{\lambda},\bar{y},\bar{x},\bar{\xi},\bar{v}\right)$ is feasible for (FD1). Since

$$\bar{v}=\frac{f(\bar{x},\bar{y})}{g(\bar{x},\bar{y})},$$

then, using the weak duality theorem (Theorem 11), we conclude that $\left(\bar{\alpha},\bar{\lambda},\bar{y},\bar{x},\bar{\xi},\bar{v}\right)$ is optimal for (FD1). Therefore, the corresponding optimal values of (FP) and (FD1) are equal. ■

Theorem 13 (*Converse Duality*). *Let $\left(\bar{\alpha},\bar{\lambda},\bar{y},\bar{u},\bar{\xi},\bar{v}\right)$ be an optimal point in* (FD1) *such that $\bar{u}\in D$. Assume that the function $\sum_{i=1}^{\bar{\alpha}}\bar{\lambda}_i\left(f(\cdot,y^i)-\bar{v}g(\cdot,y^i)\right)$ is B-(p,r)-invex at $\bar{u}$ on $D\cup U_1$ with respect to η and b satisfying $b(x,\bar{u})>0$ for all $x\in D$ and, moreover, the function $\sum_{j=1}^{k}\bar{\xi}_jh_j(\cdot)$ is B_h-(p,r)-invex at $\bar{u}$ on $D\cup U_1$ with respect to the same function η and with respect to b_h, not necessarily, equal to the function b. Then $\bar{u}$ is optimal in* (FP).

Proof. We proceed by contradiction. Suppose that $\bar{u}$ is not optimal in (FP). Then, there exists a feasible solution $\widetilde{x}$ in (FP), such that

$$\bar{v}=\sup_{y\in Y}\frac{f(\bar{u},y)}{g(\bar{x},y)}>\sup_{y\in Y}\frac{f(\widetilde{x},y)}{g(\widetilde{x},y)}.$$

Thus,

$$\sum_{i=1}^{\bar{\alpha}}\bar{\lambda}_i\left(f\left(\widetilde{x},\bar{y}^i\right)-\bar{v}g\left(\widetilde{x},\bar{y}^i\right)\right)<0.$$

From the feasibility of $\left(\bar{\alpha},\bar{\lambda},\bar{y},\bar{u},\bar{\xi},\bar{v}\right)$ in (FD1), we get

$$\sum_{i=1}^{\bar{\alpha}}\bar{\lambda}_i\left(f\left(\widetilde{x},\bar{y}^i\right)-\bar{v}g\left(\widetilde{x},\bar{y}^i\right)\right)<\sum_{i=1}^{\bar{\alpha}}\bar{\lambda}_i\left(f\left(\bar{u},\bar{y}^i\right)-\bar{v}g\left(\bar{u},\bar{y}^i\right)\right). \tag{30}$$

By assumption, $\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \left(f\left(\cdot, \overline{y}^i\right) - \overline{v}g\left(\cdot, \overline{y}^i\right)\right)$ is B-(p, r)-invex at $\overline{u}$ on D with respect to η and b. Also, from the assumption, it follows that $b(\widetilde{x}, \overline{u}) > 0$. Then, (30) implies

$$\frac{1}{r} b(\widetilde{x}, \overline{u}) \left(e^{r\left[\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \left(f(\widetilde{x}, \overline{y}^i) - \overline{v}g(\widetilde{x}, \overline{y}^i)\right) - \sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \left(f(\overline{u}, \overline{y}^i) - \overline{v}g(\overline{u}, \overline{y}^i)\right)\right]} - 1 \right) < 0.$$

Using Definition 1, the inequality above gives

$$\frac{1}{p} \left(\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \left(\nabla f(\overline{u}, \overline{y}^i) - \overline{v} \nabla g(\overline{u}, \overline{y}^i) \right) \right) \left(e^{p\eta(\widetilde{x}, \overline{u})} - \mathbf{1} \right) < 0. \tag{31}$$

From the feasibility of $\widetilde{x}$ in (FP) together with (20) and (21), we obtain

$$\sum_{j=1}^{k} \overline{\xi}_j h_j \left(\widetilde{x}\right) \leq \sum_{j=1}^{k} \overline{\xi}_j h_j \left(\overline{u}\right). \tag{32}$$

By assumption, $\sum_{j=1}^{k} \overline{\xi}_j h_j(\cdot)$ is B_h-(p, r)-invex at $\overline{u}$ on $D \cup U_1$ with respect to η and with respect to b_h. Since $b_h(\widetilde{x}, \overline{u}) > 0$, then (32) gives

$$\frac{1}{r} b_h \left(\widetilde{x}, \overline{u}\right) \left(e^{r\left(\sum_{j=1}^{k} \overline{\xi}_j h_j(\widetilde{x}) - \sum_{j=1}^{k} \overline{\xi}_j h_j(\overline{u}) \right)} - 1 \right) \leq 0.$$

Hence, by Definition 1, the inequality above implies

$$\frac{1}{p} \sum_{j=1}^{k} \overline{\xi}_j \nabla h_j(\overline{x}) \left(e^{p\eta(\widetilde{x}, \overline{u})} - \mathbf{1} \right) \leq 0. \tag{33}$$

Thus, by (31) and (33), we obtain the following inequality

$$\frac{1}{p} \left(\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \left(\nabla f(\overline{u}, \overline{y}^i) - \overline{v} \nabla g(\overline{u}, \overline{y}^i) \right) + \sum_{j=1}^{k} \overline{\xi}_j \nabla h_j(\overline{u}) \right) \left(e^{p\eta(\widetilde{x}, \overline{u})} - \mathbf{1} \right) < 0,$$

which contradicts (18). ■

Theorem 14 (*Strictly Converse Duality*). *Let $\overline{x}$ and $\left(\overline{\alpha}, \overline{\lambda}, \overline{y}, \overline{u}, \overline{\xi}, \overline{v}\right)$ be optimal points in* (FP) *and* (FD1), *respectively, and the Linear Independence Constraint Qualification (LICQ)* [25] *be satisfied at $\overline{x}$. Moreover, assume that $\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \left(f\left(\cdot, y^i\right) - \overline{v}g\left(\cdot, y^i\right)\right)$ is B-(p, r)-invex at $\overline{u}$ on $D \cup U_1$ with respect to η and b satisfying $b\left(x, \overline{u}\right) > 0$ for all $x \in D$ and, moreover, $\sum_{j=1}^{k} \overline{\xi}_j h_j(\cdot)$ is B_h-(p, r)-invex at $\overline{u}$ on $D \cup U_1$ with respect to the same function η and with respect to b_h, not necessarily, equal to the function b. Then $\overline{x} = \overline{u}$, that is, $\overline{u}$ is an optimal point in* (FP), *and $\overline{v} = \frac{f(\overline{u}, \overline{y})}{g(\overline{u}, \overline{y})}$.*

Proof. Suppose, contrary to the result, that $\overline{x} \neq \overline{u}$. From the strong duality theorem (Theorem 12), it follows that

$$\overline{v} = \sup_{y \in Y} \frac{f(\overline{x}, y)}{g(\overline{x}, y)}. \tag{34}$$

From the feasibility of $\overline{x}$ in (FP), we have

$$\sum_{j=1}^{k} \overline{\xi}_j h_j(\overline{x}) \leq 0. \tag{35}$$

By assumption, $\sum_{j=1}^{k} \overline{\xi}_j h_j(\cdot)$ is B_h-(p, r)-invex at $\overline{u}$ on $D \cup U_1$ with respect to η and with respect to b_h. Then, by Definition 1, there exists a function b_h such that $b_h\left(x, u\right) \geq 0$ for all $x \in D$ and $u \in U_1$. Hence, by (20) and (35),

$$\frac{1}{r} b_h \left(\overline{x}, \overline{u}\right) \left(e^{r\left(\sum_{j=1}^{k} \overline{\xi}_j h_j(\overline{x}) - \sum_{j=1}^{k} \overline{\xi}_j h_j(\overline{u}) \right)} - 1 \right) \leq 0.$$

Then, by Definition 1, we get

$$\frac{1}{p} \sum_{j=1}^{k} \overline{\xi}_j \nabla h_j(\overline{u}) \left(e^{p\eta(\overline{x}, \overline{u})} - \mathbf{1} \right) \leq 0. \tag{36}$$

Therefore, by (36), we obtain the inequality

$$\frac{1}{p}\sum_{i=1}^{\overline{\alpha}}\overline{\lambda}_i\left(\nabla f(\overline{u},\overline{y}^i)-\overline{v}\nabla g(\overline{u},\overline{y}^i)\right)\left(e^{p\eta(\overline{x},\overline{u})}-\mathbf{1}\right)\geq 0.$$

By assumption, $\sum_{i=1}^{\overline{\alpha}}\overline{\lambda}_i\left(f\left(\cdot,\overline{y}^i\right)-\overline{v}g\left(\cdot,\overline{y}^i\right)\right)$ is strictly B-(p,r)-invex with respect to η and b at $\overline{u}$ on $D\cup U_1$. Then, by the definition of strictly B-(p,r)-invexity (Definition 1) and (36), it follows that

$$\frac{1}{r}b\left(\overline{x},\overline{u}\right)\left(e^{r\left(\sum_{i=1}^{\overline{\alpha}}\overline{\lambda}_i\left[f(\overline{x},\overline{y}^i)-\overline{v}g(\overline{x},\overline{y}^i)\right]-\sum_{i=1}^{\overline{\alpha}}\overline{\lambda}_i\left[f(\overline{u},\overline{y}^i)-\overline{v}g(\overline{u},\overline{y}^i)\right]\right)}-1\right)>0.$$

From the hypothesis $b\left(\overline{x},\overline{u}\right)>0$, it follows that

$$\sum_{i=1}^{\overline{\alpha}}\overline{\lambda}_i\left[f(\overline{x},\overline{y}^i)-\overline{v}g(\overline{x},\overline{y}^i)\right]-\sum_{i=1}^{\overline{\alpha}}\overline{\lambda}_i\left[f(\overline{u},\overline{y}^i)-\overline{v}g(\overline{u},\overline{y}^i)\right]>0.$$

Thus, by (19),

$$\sum_{i=1}^{\overline{\alpha}}\overline{\lambda}_i\left[f(\overline{x},\overline{y}^i)-\overline{v}g(\overline{x},\overline{y}^i)\right]>0.$$

Since $\overline{\lambda}_i\geq 0$ for $i=1,\ldots,\overline{\alpha}$, therefore, there exists i^* such that

$$f(\overline{x},\overline{y}^{i^*})-\overline{v}g(\overline{x},\overline{y}^{i^*})>0.$$

Hence, we obtain the following inequality

$$\frac{f(\overline{x},\overline{y}^{i^*})}{g(\overline{x},\overline{y}^{i^*})}>\overline{v},$$

which contradicts (34). This completes the proof. ■

Remark 15. The theorem above can be proved if, in place of the strictly B-(p,r)-invexity assumption of the function $\sum_{i=1}^{\overline{\alpha}}\lambda_i\left(f\left(\cdot,\overline{y}^i\right)-\overline{v}g\left(\cdot,\overline{y}^i\right)\right)$, we assume strictly B_h-(p,r)-invexity of the function $\sum_{j=1}^{k}\overline{\xi}_jh_j(\cdot)$ (at $\overline{u}$ on $D\cup U_1$).

4.2. Weir duality

Now, on the lines of Weir [26], we consider a dual problem (FD2) to (FP) as follows:

$$\max_{(\alpha,\lambda,\overline{y})\in A}\ \sup_{(u,\xi)\in H_2(\alpha,\lambda,\overline{y})}\ \Phi(u),$$

where the set $H_2\left(\alpha,\lambda,y\right)$ is the set of all $(u,\xi)\in X\times R_+^k$ satisfying the following conditions:

$$\sum_{i=1}^{\alpha}\lambda_i\left(\nabla f(u,y^i)g(u,y^i)-f(u,y^i)\nabla g(u,y^i)\right)+\sum_{j=1}^{k}\xi_j\nabla h_j(u)=0, \tag{37}$$

(FD2)

$$\sum_{j=1}^{k}\xi_jh_j(u)\geq 0, \tag{38}$$

$$\lambda_i\geq 0,\quad \sum_{i=1}^{\alpha}\lambda_i=1,\qquad y^i\in Y\left(u\right),\quad i=1,\ldots,\alpha,\qquad \xi_j\geq 0,\quad j=1,\ldots,k, \tag{39}$$

where $\Phi(u)=\frac{f(u,y)}{g(u,y)}$.

The Weir dual problem we also define as follows:

$$\max_{(\alpha,\lambda,y)\in A}\ \sup_{(u,\xi)\in H_2(\alpha,\lambda,y)}\ \Phi(u),$$

subject to (38)– (39) and (40) in place of (37) (FD2p)

$$\left.\begin{array}{l}\frac{1}{p}\nabla\left[\sum_{i=1}^{\alpha}\lambda_i\left(\nabla f(u,y^i)g(u,y^i)-f(u,y^i)\nabla g(u,y^i)\right)+\sum_{j=1}^{k}\xi_jh_j(u)\right]\left(e^{p\eta(x,u)}-\mathbf{1}\right)\geq 0\quad \text{if } p\neq 0\\ \nabla\left[\sum_{i=1}^{\alpha}\lambda_i\left(\nabla f(u,y^i)g(u,y^i)-f(u,y^i)\nabla g(u,y^i)\right)+\sum_{j=1}^{k}\xi_jh_j(u)\right]\eta(x,u)\geq 0\quad \text{if } p=0\end{array}\right\}\ \text{for all } x\in D. \tag{40}$$

Thus, by (38)–(40), we define a class of Weir dual problems with respect to the function η and the scalar p.

If, for a triplet $(\alpha,\lambda,\overline{y})\in A$, the set $H_2\left(\alpha,\lambda,\overline{y}\right)$ is empty, then we define the supremum over it to be $-\infty$.

Let W_{FD2} denote a set of all feasible solutions for problem (FD2). Moreover, let U_2 denote

$$U_2 = \{u \in X : (\alpha, \lambda, \bar{y}, u, \xi) \in W_{FD2}\}.$$

Theorem 16 (*Weak Duality*). *Let x and $(\alpha, \lambda, y, u, \xi)$ be feasible solutions for* (FP) *and for* (FD2), *respectively. If $\sum_{i=1}^{\alpha} \lambda_i[f(\cdot, y^i)g(u, y^i) - f(u, y^i)g(\cdot, y^i)]$ is B-(p, r)-invex at u on $D \cup U_2$ with respect to η and with respect to b satisfying $b(x, u) > 0$ and, moreover, $\sum_{j=1}^{k} \bar{\xi}_j h_j(\cdot)$ is B_h-(p, r)-invex at u on $D \cup U_2$ with respect to the same function η and with respect to b_h, not necessarily, equal to the function b. Then*

$$\sup_{y \in Y} \frac{f(x, y)}{g(x, y)} \geq \Phi(u).$$

Proof. We denote

$$\Psi(x) = \sum_{i=1}^{\alpha} \lambda_i \left[f(x, y^i)g(u, y^i) - f(u, y^i)g(x, y^i) \right]. \tag{41}$$

From the feasibility of x in (FP) together with $\xi_j \geq 0, j = 1, \ldots k$, we have

$$\sum_{j=1}^{k} \xi_j h_j(x) \leq 0. \tag{42}$$

By assumption, $\sum_{j=1}^{k} \xi_j h_j(\cdot)$ is B_h-(p, r)-invex with respect to η and b_h at u on $D \cup U_2$. Hence, by (38) and (42),

$$\frac{1}{r} b_h(x, u) \left(e^{r\left(\sum_{j=1}^{k} \xi_j h_j(x) - \sum_{j=1}^{k} \xi_j h_j(u) \right)} - 1 \right) \leq 0.$$

Then by Definition 1 we conclude that

$$\frac{1}{p} \sum_{j=1}^{k} \xi_j \nabla h_j(u) \left(e^{p\eta(x,u)} - \mathbf{1} \right) \leq 0.$$

Thus, by the first constraint of (FD2),

$$\frac{1}{p} \sum_{i=1}^{\alpha} \lambda_i \left[\nabla f(u, y^i)g(u, y^i) - f(u, y^i)\nabla g(u, y^i) \right] \left(e^{p\eta(x,u)} - \mathbf{1} \right) \geq 0. \tag{43}$$

Hence, (41) yields

$$\frac{1}{p} \sum_{i=1}^{\alpha} \lambda_i \nabla \Psi(u) \left(e^{p\eta(x,u)} - \mathbf{1} \right) \geq 0. \tag{44}$$

By assumption, Ψ is B-(p, r)-invex at u on $D \cup U_2$ with respect to η and b. Then, by Definition 1, (44) gives

$$\frac{1}{r} b(x, u) \left(e^{r(\Psi(x) - \Psi(u))} - 1 \right) \geq 0.$$

Since $b(x, u) > 0$ for all $x, u \in D \cup U_2$, then by (41), we get

$$\Psi(x) \geq \Psi(u). \tag{45}$$

By (41), $\Psi(u) = 0$. Thus, by (41) and (45), it follows that

$$\sum_{i=1}^{\alpha} \lambda_i \left[f(x, y^i)g(u, y^i) - f(u, y^i)g(x, y^i) \right] \geq 0.$$

Since $\bar{\lambda}_i \geq 0$ for $i = 1, \ldots, \alpha$, therefore, there exists i^* such that

$$f(x, y^{i^*})g(u, y^{i^*}) - f(u, y^{i^*})g(x, y^{i^*}) \geq 0. \tag{46}$$

By (46), we have

$$\sup_{y \in Y} \frac{f(x, y)}{g(x, y)} \geq \frac{f(x, y^{i^*})}{g(x, y^{i^*})} \geq \frac{f(u, y^{i^*})}{g(u, y^{i^*})}.$$

Since $y^{i^*} \in Y(u)$, then

$$\Phi(u) = \frac{f(u, y^{i^*})}{g(u, y^{i^*})}.$$

This completes the proof. ■

Theorem 17 (*Strong Duality*). *Let* $\bar{x}$ *be an optimal point in* (FP) *and the Linear Independence Constraint Qualification (LICQ)* [25] *be satisfied at* $\bar{x}$. *Then, there exist* $\left(\bar{\alpha}, \bar{\lambda}, \bar{y}\right) \in A$ *and* $(\bar{x}, \bar{\xi}) \in H\left(\bar{\alpha}, \bar{\lambda}, \bar{y}\right)$ *such that* $\left(\bar{\alpha}, \bar{\lambda}, \bar{y}, \bar{x}, \bar{\xi}\right)$ *is optimal in* (FD2). *If also the hypotheses of Theorem 16 hold, then the corresponding optimal values of* (FP) *and* (FD2) *are equal.*

Proof. By assumption, $\bar{x}$ is an optimal point of (FP) and the Linear Independence Constraint Qualification (LICQ) holds at $\bar{x}$. Then, by conditions (15)–(17), we conclude that $\left(\bar{\alpha}, \bar{\lambda}, \bar{y}, \bar{x}, \bar{\xi}\right)$ is feasible for (FD2). Since

$$\Phi(\bar{x}) = \sup_{y \in Y} \frac{f(\bar{x}, y)}{g(\bar{x}, y)},$$

then, using the weak duality theorem (Theorem 16), we conclude that $\left(\bar{\alpha}, \bar{\lambda}, \bar{y}, \bar{x}, \bar{\xi}\right)$ is optimal for (FD2). Therefore, the corresponding optimal values of (FP) and (FD2) are equal. ■

Theorem 18 (*Converse Duality*). *Let* $\left(\bar{\alpha}, \bar{\lambda}, \bar{y}, \bar{u}, \bar{\xi}\right)$ *be an optimal point in* (FD2) *such that* $\bar{u} \in D$. *Assume that* $\sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i \left(f\left(\bar{u}, y^i\right) g\left(\cdot, y^i\right) - f\left(\cdot, y^i\right) g\left(\bar{u}, y^i\right)\right)$ *is* B-(p, r)*-invex function at* $\bar{u}$ *on* $D \cup U_2$ *with respect to* η *and* b *satisfying* $b(x, \bar{u}) > 0$ *for all* $x \in D$ *and* $\sum_{j=1}^{k} \bar{\xi}_j h_j(\cdot)$ *is* B_h-(p, r)*-invex function at* $\bar{u}$ *on* $D \cup U_2$ *with respect to the same function* η *and with respect to* b_h, *not necessarily, equal to the function* b. *Then* $\bar{u}$ *is optimal in* (FP).

Theorem 19 (*Strictly Converse Duality*). *Let* $\bar{x}$ *and* $\left(\bar{u}, \bar{\xi}, \bar{\alpha}, \bar{\lambda}, \bar{y}\right)$ *be optimal points in* (FP) *and* (FD2), *respectively, and the Linear Independence Constraint Qualification (LICQ) be satisfied at* $\bar{x}$. *Moreover, we assume that* $\sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i (f(\cdot, y^i) g(\bar{u}, y^i) - f(\bar{u}, y^i) g(\cdot, y^i))$ *is strictly* B-(p, r)*-invex at* $\bar{u}$ *on* $D \cup U_2$ *with respect to* η *and* b, *and, moreover,* $\sum_{j=1}^{k} \bar{\xi}_j h_j(\cdot)$ *is* B_h-(p, r)*-invex at* $\bar{u}$ *on* $D \cup U_2$ *with respect to the same function* η *and with respect to* b_h, *not necessarily equal to* b. *Then* $\bar{x} = \bar{u}$, *that is,* $\bar{u}$ *is an optimal point in* (FP).

Proof. Suppose, contrary to the result, that $\bar{x} \neq \bar{u}$. From the feasibility of $\bar{x}$ in (FP) together by (39), it follows that

$$\sum_{j=1}^{k} \xi_j h_j(\bar{x}) \leq 0. \tag{47}$$

By assumption, $\sum_{j=1}^{k} \bar{\xi}_j h_j(\cdot)$ is B_h-(p, r)-invex at $\bar{u}$ on $D \cup U_2$ with respect to η and with respect to b_h. Thus, by (38) and (47), it follows that

$$\frac{1}{r} b_h(\bar{x}, \bar{u}) \left(e^{r\left(\sum_{j=1}^{k} \bar{\xi}_j h_j(\bar{x}) - \sum_{j=1}^{k} \bar{\xi}_j h_j(\bar{u})\right)} - 1\right) \leq 0.$$

Then, by Definition 1, we get

$$\frac{1}{p} \sum_{j=1}^{k} \bar{\xi}_j \nabla h_j(\bar{u}) \left(e^{p\eta(\bar{x}, \bar{u})} - \mathbf{1}\right) \leq 0.$$

Hence, by (37),

$$\frac{1}{p} \sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i \left(\nabla f(\bar{u}, \bar{y}^i) g(\bar{u}, \bar{y}^i) - f(\bar{u}, \bar{y}^i) \nabla g(\bar{u}, \bar{y}^i)\right) \left(e^{p\eta(\bar{x}, \bar{u})} - \mathbf{1}\right) \geq 0.$$

By assumption, $\sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i \left(f\left(\cdot, \bar{y}^i\right) g(\bar{u}, \bar{y}^i) - f(\bar{u}, \bar{y}^i) g\left(\cdot, \bar{y}^i\right)\right)$ is strictly B-(p, r)-invex with respect to η and b at u on $D \cup U_2$. Then, by the definition of strictly B-(p, r)-invexity (Definition 1) and the inequality above, it follows that

$$\frac{1}{r} b(\bar{x}, \bar{u}) \left(e^{r\left(\sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i \left[f(\bar{x}, \bar{y}^i) g(\bar{u}, \bar{y}^i) - f(\bar{u}, \bar{y}^i) g(\bar{x}, \bar{y}^i)\right] - \sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i \left[f(\bar{u}, \bar{y}^i) g(\bar{u}, \bar{y}^i) - f(\bar{u}, \bar{y}^i) g(\bar{u}, \bar{y}^i)\right]\right)} - 1\right) > 0.$$

Thus,

$$\sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i \left[f(\bar{x}, \bar{y}^i) g(\bar{u}, \bar{y}^i) - f(\bar{u}, \bar{y}^i) g(\bar{x}, \bar{y}^i)\right] \geq 0.$$

Since $\bar{\lambda}_i \geq 0$ for $i = 1, \ldots, \alpha$, therefore, there exists i^* such that

$$f(\bar{x}, \bar{y}^{i^*}) g(\bar{u}, \bar{y}^{i^*}) - f(\bar{u}, \bar{y}^{i^*}) g(\bar{x}, \bar{y}^{i^*}) > 0.$$

Hence, we obtain the following inequality

$$\frac{f(\bar{x}, \bar{y}^{i^*})}{g(\bar{x}, \bar{y}^{i^*})} > \frac{f(\bar{u}, \bar{y}^{i^*})}{g(\bar{u}, \bar{y}^{i^*})}.$$

Thus,

$$\sup_{y\in Y}\frac{f(\overline{x},y)}{g(\overline{x},y)}\geq\frac{f(\overline{x},y^{i^*})}{g(\overline{x},y^{i^*})}>\frac{f(\overline{u},y^{i^*})}{g(\overline{u},y^{i^*})}.$$

Since $y^{i^*}\in Y(\overline{u})$, then

$$\Phi(\overline{u})=\frac{f(\overline{u},y^{i^*})}{g(\overline{u},y^{i^*})},$$

which contradicts the strong duality theorem (Theorem 12). This completes the proof. ■

Remark 20. The theorem above can be proved if, in place of the assumption of strictly B-(p,r)-invexity of the function $\sum_{i=1}^{\overline{\alpha}}\overline{\lambda}_i\left(f\left(\cdot,\overline{y}^i\right)g(\overline{u},\overline{y}^i)-f(\overline{u},\overline{y}^i)g\left(\cdot,\overline{y}^i\right)\right)$, we assume strictly B_h-(p,r)-invexity of the function $\sum_{j=1}^{k}\overline{\xi}_jh_j(\cdot)$ (at $\overline{u}$ on $D\cup U_2$).

4.3. Bector duality

Now, in relation to (FP) consider the following analogue of Bector's dual:

$$\max_{(\alpha,\lambda,\overline{y})\in A}\ \sup_{(u,\xi)\in H_3(\alpha,\lambda,\overline{y})}\frac{\sum_{i=1}^{\alpha}\lambda_if\left(u,y^i\right)}{\sum_{i=1}^{\alpha}\lambda_ig\left(u,y^i\right)},$$

where $H_3(\alpha,\lambda,y)$ is the set of all $(u,\xi)\in X\times R_+^k$ satisfying the following conditions:

$$G(u)\sum_{i=1}^{\alpha}\lambda_i\nabla f\left(u,y^i\right)+F(u)\sum_{i=1}^{\alpha}\lambda_i\nabla g\left(u,y^i\right)+\sum_{j=1}^{k}\xi_j\nabla h_j(u)=0,\tag{48}$$

(FD3)

$$\sum_{j=1}^{k}\xi_jh_j(u)\geq 0,\tag{49}$$

$$\lambda_i\geq 0,\quad \sum_{i=1}^{\alpha}\lambda_i=1,\qquad y^i\in Y\left(u\right),\quad i=1,\ldots,\alpha,\qquad \xi_j\geq 0,\quad j=1,\ldots,k,\tag{50}$$

where, for the sake of convenience, we use the following denotations:

$$F(u):=\sum_{i=1}^{\alpha}\lambda_if\left(u,y^i\right),\tag{51}$$

$$G(u):=\sum_{i=1}^{\alpha}\lambda_ig\left(u,y^i\right).\tag{52}$$

The Bector dual problem (FD3) we also define as follows:

$$\max_{(\alpha,\lambda,\overline{y})\in A}\ \sup_{(u,\xi)\in H_3(\alpha,\lambda,\overline{y})}\frac{\sum_{i=1}^{\alpha}\lambda_if\left(u,y^i\right)}{\sum_{i=1}^{\alpha}\lambda_ig\left(u,y^i\right)},$$

subject to (49)–(50) and (53) in place of (48) (FD3p)

$$\begin{array}{ll}\frac{1}{p}\left[G(u)\sum_{i=1}^{\alpha}\lambda_i\nabla f\left(u,y^i\right)+F(u)\sum_{i=1}^{\alpha}\lambda_i\nabla g\left(u,y^i\right)+\sum_{j=1}^{k}\xi_j\nabla h_j(u)\right]\left(e^{p\eta(x,u)}-\mathbf{1}\right)\geq 0\quad \text{if } p\neq 0 & \\ \left[G(u)\sum_{i=1}^{\alpha}\lambda_i\nabla f\left(u,y^i\right)+F(u)\sum_{i=1}^{\alpha}\lambda_i\nabla g\left(u,y^i\right)+\sum_{j=1}^{k}\xi_j\nabla h_j(u)\right]\eta(x,u)\geq 0\quad \text{if } p=0 & \end{array}\quad \text{for all } x\in D.\tag{53}$$

Thus, by (49)–(50) and (53), we define a class of Bector dual problems with respect to the function η and the scalar p.
If, for a triplet $(\alpha,\lambda,\overline{y})\in A$, the set $H_3(\alpha,\lambda,\overline{y})$ is empty, then we define the supremum over it to be $-\infty$.
Let W_{FD3} denote a set of all feasible solutions for problem (FD3). Moreover, let U_3 denote

$$U_3=\{u\in X:(\alpha,\lambda,\overline{y},u,\xi)\in W_{FD3}\}.$$

Theorem 21 (*Weak Duality*). *Let x and $(\alpha, \lambda, y, u, \xi)$ be feasible solutions for* (FP) *and* (FD3), *respectively. Further, we assume that:*

(a) *$G(u)\sum_{i=1}^{\alpha}\lambda_i f\left(\cdot, y^i\right)$ is B-(p,r)-invex at u on $D\cup U_3$ with respect to η and b;*
(b) *$F(u)\sum_{i=1}^{\alpha}\lambda_i g\left(\cdot, y^i\right)$ is B-$(-p,r)$-incave at u on $D\cup U_3$ with respect to $-\eta$ and b;*
(c) *$\sum_{j=1}^{k}\xi_j h_j(\cdot)$ is B_h-(p,r)-invex at u on $D\cup U_3$ with respect to η and with respect to b_h, not necessarily equal to b.*

If $b\,(x,u) > 0$, then

$$\sup_{y\in Y}\frac{f(x,y)}{g(x,y)} \geq \frac{\sum_{i=1}^{\alpha}\lambda_i f\left(u, y^i\right)}{\sum_{i=1}^{\alpha}\lambda_i g\left(u, y^i\right)}.$$

Proof. By assumption, $\sum_{j=1}^{k}\xi_j h_j(\cdot)$ is B_h-(p,r)-invex at u on $D\cup U_3$ with respect to η and with respect to b_h. Then, by Definition 1,

$$\frac{1}{r}b_h\,(x,u)\left(\mathrm{e}^{r\left(\sum_{j=1}^{k}\xi_j h_j(x)-\sum_{j=1}^{k}\xi_j h_j(u)\right)}-1\right) \geq \frac{1}{p}\sum_{j=1}^{k}\xi_j\nabla h_j(u)\left(\mathrm{e}^{p\eta(x,u)}-\mathbf{1}\right).$$

Using $x\in D$ together with (49) and (50), we get

$$\frac{1}{p}\sum_{j=1}^{k}\xi_j\nabla h_j(u)\left(\mathrm{e}^{p\eta(x,u)}-\mathbf{1}\right)\leq 0.$$

Then, (48) gives,

$$\frac{1}{p}\left[G(u)\sum_{i=1}^{\alpha}\lambda_i\nabla f\left(u,y^i\right)+F(u)\sum_{i=1}^{\alpha}\lambda_i\nabla g\left(u,y^i\right)\right]\left(\mathrm{e}^{p\eta(x,u)}-\mathbf{1}\right)\geq 0. \tag{54}$$

By assumption, $G(u)\sum_{i=1}^{\alpha}\lambda_i f\left(\cdot, y^i\right)$ is B-(p,r)-invex at u on $D\cup U_3$ with respect to η and b satisfying $b(x,u)>0$. Thus,

$$\frac{1}{r}b\,(x,u)\left(\mathrm{e}^{rG(u)\sum_{i=1}^{\alpha}\lambda_i\left[f\left(x,y^i\right)-f\left(u,y^i\right)\right]}-1\right)\geq\frac{1}{p}G(u)\sum_{i=1}^{\alpha}\lambda_i\nabla f\left(u,y^i\right)\left(\mathrm{e}^{p\eta(x,u)}-\mathbf{1}\right). \tag{55}$$

By assumption, $F(u)\sum_{i=1}^{\alpha}\lambda_i g\left(\cdot, y^i\right)$ is B-$(-p,r)$-incave with respect to $-\eta$ and b at u on $D\cup U_3$. Then, by Remark 4,

$$\frac{1}{r}b\,(x,u)\left(\mathrm{e}^{rF(u)\sum_{i=1}^{\alpha}\lambda_i\left[g\left(x,y^i\right)-g\left(u,y^i\right)\right]}-1\right)\leq\frac{1}{(-p)}F(u)\sum_{i=1}^{\alpha}\lambda_i\nabla g\left(u,y^i\right)\left(\mathrm{e}^{-p[-\eta(x,u)]}-\mathbf{1}\right). \tag{56}$$

Thus, by (54)–(56),

$$\frac{1}{r}b\,(x,u)\left(\mathrm{e}^{rG(u)\sum_{i=1}^{\alpha}\lambda_i\left[f\left(x,y^i\right)-f\left(u,y^i\right)\right]}-1\right)\geq\frac{1}{r}b\,(x,u)\left(\mathrm{e}^{rF(u)\sum_{i=1}^{\alpha}\lambda_i\left[g\left(x,y^i\right)-g\left(u,y^i\right)\right]}-1\right).$$

Therefore,

$$G(u)\sum_{i=1}^{\alpha}\lambda_i\left[f\left(x,y^i\right)-f\left(u,y^i\right)\right]\geq F(u)\sum_{i=1}^{\alpha}\lambda_i\left[g\left(x,y^i\right)-g\left(u,y^i\right)\right].$$

Using (51) and (52), we obtain

$$\sum_{i=1}^{\alpha}\lambda_i g\left(u,y^i\right)\sum_{i=1}^{\alpha}\lambda_i\left[f\left(x,y^i\right)-f\left(u,y^i\right)\right]\geq\sum_{i=1}^{\alpha}\lambda_i f\left(u,y^i\right)\sum_{i=1}^{\alpha}\lambda_i\left[g\left(x,y^i\right)-g\left(u,y^i\right)\right].$$

Thus,

$$\sum_{i=1}^{\alpha}\lambda_i f\left(x,y^i\right)\sum_{i=1}^{\alpha}\lambda_i g\left(u,y^i\right)\geq\sum_{i=1}^{\alpha}\lambda_i f\left(u,y^i\right)\sum_{i=1}^{\alpha}\lambda_i g\left(x,y^i\right),$$

so that,

$$\frac{\sum_{i=1}^{\alpha}\lambda_i f\left(x,y^i\right)}{\sum_{i=1}^{\alpha}\lambda_i g\left(x,y^i\right)}\geq\frac{\sum_{i=1}^{\alpha}\lambda_i f\left(u,y^i\right)}{\sum_{i=1}^{\alpha}\lambda_i g\left(u,y^i\right)}.$$

Since $\lambda_i \geq 0$, $i = 1, \ldots, \alpha$, therefore, there exists i^* such that

$$\frac{f\left(x, y^{i^*}\right)}{g\left(x, y^{i^*}\right)} \geq \frac{\sum_{i=1}^{\alpha} \lambda_i f\left(u, y^i\right)}{\sum_{i=1}^{\alpha} \lambda_i g\left(u, y^i\right)}.$$

Therefore,

$$\sup_{y \in Y} \frac{f(x, y)}{g(x, y)} \geq \frac{\sum_{i=1}^{\alpha} \lambda_i f\left(u, y^i\right)}{\sum_{i=1}^{\alpha} \lambda_i g\left(u, y^i\right)}.$$

This means that the weak duality in the sense of Bector holds between (FP) and (FD3) and completes the proof of theorem. ■

Theorem 22 (*Strong Duality*). *Let $\bar{x}$ be an optimal point in* (FP) *and the Linear Independence Constraint Qualification (LICQ)* [25] *be satisfied at $\bar{x}$. Then, there exist $\left(\bar{\alpha}, \bar{\lambda}, \bar{y}\right) \in A$ and $(\bar{x}, \bar{\xi}) \in H_3(\bar{\alpha}, \bar{\lambda}, \bar{y})$ such that $\left(\bar{\alpha}, \bar{\lambda}, \bar{y}, \bar{x}, \bar{\xi}\right)$ is an optimal point in* (FD3). *If also the hypotheses of Theorem 21 hold, then the corresponding optimal values of* (FP) *and* (FD3) *are equal.*

Proof. By assumption, $\bar{x}$ is an optimal point of (FP) and the Linear Independence Constraint Qualification (LICQ) holds at $\bar{x}$. Then, by (15)–(17), we conclude that $\left(\bar{x}, \bar{\xi}, \bar{\alpha}, \bar{\lambda}, \bar{y}\right)$ is feasible for (FD3). Since

$$\frac{\sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i f\left(\bar{x}, \bar{y}^i\right)}{\sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i g\left(\bar{x}, \bar{y}^i\right)} = \sup_{y \in Y} \frac{f(\bar{x}, y)}{g(\bar{x}, y)},$$

then, using the weak duality theorem (Theorem 21), it follows that $\left(\bar{x}, \bar{\xi}, \bar{\alpha}, \bar{\lambda}, \bar{y}\right)$ is optimal for (FD3). Therefore, the corresponding optimal values of (FP) and (FD3) are equal. ■

Theorem 23 (*Converse Duality*). *Let $\left(\bar{\alpha}, \bar{\lambda}, \bar{y}, \bar{u}, \bar{\xi}\right)$ be an optimal point in* (FD3) *such that $\bar{u} \in D$. Further, assume:*

(a) *$G(\bar{u}) \sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i f\left(\cdot, y^i\right)$ is B-(p, r)-invex at $\bar{u}$ on $D \cup U_3$ with respect to η and b;*
(b) *$F(\bar{u}) \sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i g\left(\cdot, y^i\right)$ is B-$(-p, r)$-incave at $\bar{u}$ on $D \cup U_3$ with respect to $-\eta$ and b;*
(c) *$\sum_{j=1}^{k} \bar{\xi}_j h_j(\cdot)$ is B-(p, r)-invex at $\bar{u}$ on $D \cup U_3$ with respect to η and with respect to b_h, not necessarily equal to b.*

If $b\left(x, \bar{u}\right) > 0$ for all $x \in D$ then $\bar{u}$ is optimal in (FP).

Proof. By means of contradiction, we suppose that there exists $\tilde{x} \in D$ such that

$$\sup_{y \in Y} \frac{f(x, y)}{g(x, y)} < \frac{\sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i f\left(\bar{u}, y^i\right)}{\sum_{i=1}^{\alpha} \bar{\lambda}_i g\left(\bar{u}, y^i\right)}.$$

Thus,

$$\sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i f\left(\tilde{x}, y^i\right) \sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i g\left(\bar{u}, y^i\right) < \sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i f\left(\bar{u}, y^i\right) \sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i g\left(\tilde{x}, y^i\right)$$

so that,

$$\sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i g\left(\bar{u}, y^i\right) \sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i \left[f\left(\tilde{x}, y^i\right) - f\left(\bar{u}, y^i\right)\right] < \sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i f\left(\bar{u}, y^i\right) \sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i \left[g\left(\tilde{x}, y^i\right) - g\left(\bar{u}, y^i\right)\right].$$

Thus, by (51) and (52),

$$\frac{1}{r} b\left(\tilde{x}, \bar{u}\right) \left(e^{rG(\bar{u}) \sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i \left[f\left(\tilde{x}, y^i\right) - f\left(\bar{u}, y^i\right)\right]} - 1\right) < \frac{1}{r} b\left(\tilde{x}, \bar{u}\right) \left(e^{rF(\bar{u}) \sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i \left[g\left(\tilde{x}, y^i\right) - g\left(\bar{u}, y^i\right)\right]} - 1\right). \tag{57}$$

By assumption, $G(\bar{u}) \sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i f\left(\cdot, y^i\right)$ is B-(p, r)-invex at $\bar{u}$ on $D \cup U_3$ with respect to η and b. Since $b(\tilde{x}, \bar{u}) > 0$ then, by Definition 1, we get

$$\frac{1}{r} b\left(\tilde{x}, \bar{u}\right) \left(e^{rG(\bar{u}) \sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i \left[f\left(\tilde{x}, y^i\right) - f\left(\bar{u}, y^i\right)\right]} - 1\right) \geq \frac{1}{p} G(\bar{u}) \sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i \nabla f\left(\bar{u}, y^i\right) \left(e^{p\eta(\tilde{x}, \bar{u})} - \mathbf{1}\right). \tag{58}$$

By assumption, $F(\overline{u}) \sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i g\left(\cdot, y^i\right)$ is B-$(-p, r)$-incave with respect to $-\eta$ and b at $\overline{u}$ on $D \cup U_3$. Then, by Remark 4,

$$\frac{1}{r} b\left(\widetilde{x}, \overline{u}\right)\left(\mathrm{e}^{rF(u) \sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i\left[g\left(\widetilde{x}, y^i\right)-g\left(\overline{u}, y^i\right)\right]}-1\right) \leq \frac{1}{(-p)} F(\overline{u}) \sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \nabla g\left(\overline{u}, y^i\right)\left(\mathrm{e}^{-p[-\eta(\widetilde{x}, \overline{u})]}-\mathbf{1}\right). \tag{59}$$

After multiplying (59) by -1, we add both sides of (58) and (59). Then, using (57), we get

$$\frac{1}{p}\left[G(\overline{u}) \sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \nabla f\left(\overline{u}, y^i\right)+F(\overline{u}) \sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \nabla g\left(\overline{u}, y^i\right)\right]\left(\mathrm{e}^{p \eta(\widetilde{x}, \overline{u})}-\mathbf{1}\right)<0. \tag{60}$$

From the feasibility of $\widetilde{x}$ in (FP), by (50), we obtain

$$\sum_{j=1}^{k} \overline{\xi}_j h_j(\widetilde{x}) \leq 0. \tag{61}$$

By assumption, $\sum_{j=1}^{k} \overline{\xi}_j h_j(\cdot)$ is B_h-(p, r)-invex at $\overline{u}$ on $D \cup U_3$ with respect to η and with respect to b_h, not necessarily equal to b. Then, by Definition 1, there exists a function b_h such that $b_h\left(\widetilde{x}, \overline{u}\right)>0$. Hence, by (49) and (61), it follows that

$$\frac{1}{r} b_h\left(\widetilde{x}, \overline{u}\right)\left(\mathrm{e}^{r\left(\sum_{j=1}^{k} \overline{\xi}_j h_j(\widetilde{x})-\sum_{j=1}^{k} \overline{\xi}_j h_j(\overline{u})\right)}-1\right) \leq 0.$$

Then by Definition 1 we conclude that

$$\frac{1}{p} \sum_{j=1}^{k} \overline{\xi}_j \nabla h_j(\overline{u})\left(\mathrm{e}^{p \eta(\widetilde{x}, \overline{u})}-\mathbf{1}\right) \leq 0. \tag{62}$$

Hence, by (60) and (62), we obtain the inequality

$$\frac{1}{p}\left[G(\overline{u}) \sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \nabla f\left(\overline{u}, y^i\right)+F(\overline{u}) \sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \nabla g\left(\overline{u}, y^i\right)+\sum_{j=1}^{k} \overline{\xi}_j \nabla h_j(\overline{u})\right]\left(\mathrm{e}^{p \eta(\widetilde{x}, \overline{u})}-\mathbf{1}\right)<0,$$

which contradicts (48). This completes the proof. ■

Theorem 24 (*Strictly Converse Duality*). *Let $\overline{x}$ and $\left(\overline{\alpha}, \overline{\lambda}, \overline{y}, \overline{u}, \overline{\xi}\right)$ be an optimal points in* (FP) *and* (FD3) *and the Linear Independence Constraint Qualification (LICQ)* [25] *be satisfied at $\overline{x}$. Further, assume:*

(a) *$G(\overline{u}) \sum_{i=1}^{\alpha} \overline{\lambda}_i f\left(\cdot, y^i\right)$ is B-(p, r)-invex at $\overline{u}$ on $D \cup U_3$ with respect to η and b;*
(b) *$F(\overline{u}) \sum_{i=1}^{\alpha} \overline{\lambda}_i g\left(\cdot, y^i\right)$ is B-$(-p, r)$-incave at $\overline{u}$ on $D \cup U_3$ with respect to $-\eta$ and b;*
(c) *$\sum_{j=1}^{k} \overline{\xi}_j h_j(\cdot)$ is strictly B_h-(p, r)-invex at $\overline{u}$ on $D \cup U_3$ with respect to η and with respect to b_h, not necessarily equal to b.*
If $b\left(\overline{x}, \overline{u}\right)>0$ then $\overline{x}=\overline{u}$, that is, $\overline{u}$ is an optimal point in (FP).

Proof. Proof of this theorem is similar to the proof of Theorem 21 and, therefore, it was omitted in the paper. ■

Following the approaches of Bector et al. [12], we formulate the following dual problem for (FP)

$$\max_{(\alpha, \lambda, \overline{y}) \in A} \sup_{(u, \xi) \in H_4(\alpha, \lambda, \overline{y})} \frac{\sum_{i=1}^{\alpha} \lambda_i f\left(u, y^i\right)+\sum_{j=1}^{k} \xi_j h_j(u)}{\sum_{i=1}^{\alpha} \lambda_i g\left(u, y^i\right)},$$

where $H_4(\alpha, \lambda, y)$ as the set of all $(u, \xi) \in X \times R_{+}^{k}$ satisfying the following conditions:

$$\nabla\left[\frac{\sum_{i=1}^{\alpha} \lambda_i f\left(u, y^i\right)+\sum_{j=1}^{k} \xi_j h_j(u)}{\sum_{i=1}^{\alpha} \lambda_i g\left(u, y^i\right)}\right]=0, \tag{63}$$

(FD4)

$$\lambda_i \geq 0, \quad \sum_{i=1}^{\alpha} \lambda_i=1, \quad y^i \in Y(u), \quad i=1, \ldots, \alpha, \quad \xi_j \geq 0, \quad j=1, \ldots, k, \tag{64}$$

We also define the Bector dual problem (FD4) as follows:

$$\max_{(\alpha,\lambda,\overline{y})\in A} \sup_{(u,\xi)\in H_4(\alpha,\lambda,\overline{y})} \frac{\sum_{i=1}^{\alpha}\lambda_i f\left(u,y^i\right)+\sum_{j=1}^{k}\xi_j h_j\left(u\right)}{\sum_{i=1}^{\alpha}\lambda_i g\left(u,y^i\right)},$$

subject to (64) and (65) in place of (63) (FD4p)

$$\left.\begin{array}{l} \frac{1}{p}\nabla\left[\frac{\sum_{i=1}^{\alpha}\lambda_i f\left(u,y^i\right)+\sum_{j=1}^{k}\xi_j h_j\left(u\right)}{\sum_{i=1}^{\alpha}\lambda_i g\left(u,y^i\right)}\right]\left(e^{p\eta(x,u)}-\mathbf{1}\right)\geq 0 \quad \text{if } p\neq 0 \\ \nabla\left[\frac{\sum_{i=1}^{\alpha}\lambda_i f\left(u,y^i\right)+\sum_{j=1}^{k}\xi_j h_j\left(u\right)}{\sum_{i=1}^{\alpha}\lambda_i g\left(u,y^i\right)}\right]\eta(x,u)\geq 0 \quad \text{if } p=0 \end{array}\right\} \text{for all } x\in D. \tag{65}$$

Analogously, as in the proceeding case of Bector type duality, by (64) and (65), we define a class of Bector dual problems with respect to the function η and the scalar p.

If for a triplet $(\alpha,\lambda,\overline{y})\in A$ the set $H_4(\alpha,\lambda,\overline{y})$ is empty, then we define the supremum over it to be $-\infty$.

Let W_{FD4} denote a set of all feasible solutions for problem (FD4). Moreover, let U_4 denote

$$U_4=\{u\in X:(\alpha,\lambda,\overline{y},u,\xi)\in W_{FD4}\}.$$

We denote

$$\Psi(x)=\sum_{i=1}^{\alpha}\lambda_i g\left(u,y^i\right)\left[\sum_{i=1}^{\alpha}\lambda_i f\left(x,y^i\right)+\sum_{j=1}^{k}\xi_j h_j\left(x\right)\right]-\sum_{i=1}^{\alpha}\lambda_i g\left(x,y^i\right)\left[\sum_{i=1}^{\alpha}\lambda_i f\left(u,y^i\right)+\sum_{j=1}^{k}\xi_j h_j\left(u\right)\right],\quad u\in D\cup U_4. \tag{66}$$

Theorem 25 (*Weak Duality*). *Let x and (α,λ,y,u,ξ) be feasible solutions for* (FP) *and for* (FD4), *respectively. If Ψ is B-(p,r)-invex at u on $D\cup U_4$ with respect to η and b satisfying $b\left(x,u\right)>0$, then*

$$\sup_{y\in Y}\frac{f(x,y)}{g(x,y)}\geq\frac{\sum_{i=1}^{\alpha}\lambda_i f\left(u,y^i\right)+\sum_{j=1}^{k}\xi_j h_j\left(u\right)}{\sum_{i=1}^{\alpha}\lambda_i g\left(u,y^i\right)}.$$

Proof. By assumption, Ψ is B-(p,r)-invex at u on $D\cup U_4$ with respect to η and b. Then, by Definition 1, we have

$$\frac{1}{r}b\left(x,u\right)\left(e^{r(\Psi(x)-\Psi(u))}-1\right)\geq\frac{1}{p}\nabla\Psi(u)\left(e^{p\eta(x,u)}-\mathbf{1}\right).$$

From the feasibility of (α,λ,y,u,ξ) in (FD4), it follows that

$$\frac{1}{r}b\left(x,u\right)\left(e^{r(\Psi(x)-\Psi(u))}-1\right)\geq 0.$$

Since $b\left(x,u\right)>0$ for all $x\in D$ then the inequality above gives

$$\Psi(x)\geq\Psi(u).$$

By definition of Ψ, it follows that $\Psi(u)=0$. Thus,

$$\sum_{i=1}^{\alpha}\lambda_i g\left(u,y^i\right)\left[\sum_{i=1}^{\alpha}\lambda_i f\left(x,y^i\right)+\sum_{j=1}^{k}\xi_j h_j\left(x\right)\right]-\sum_{i=1}^{\alpha}\lambda_i g\left(x,y^i\right)\left[\sum_{i=1}^{\alpha}\lambda_i f\left(u,y^i\right)+\sum_{j=1}^{k}\xi_j h_j\left(u\right)\right]\geq 0.$$

From the feasibility of x in (FD4), we have $\sum_{j=1}^{k}\xi_j h_j\left(x\right)\leq 0$. Hence, the inequality

$$\sum_{i=1}^{\alpha}\lambda_i g\left(u,y^i\right)\sum_{i=1}^{\alpha}\lambda_i f\left(x,y^i\right)\geq\sum_{i=1}^{\alpha}\lambda_i g\left(x,y^i\right)\left[\sum_{i=1}^{\alpha}\lambda_i f\left(u,y^i\right)+\sum_{j=1}^{k}\xi_j h_j\left(u\right)\right]$$

holds for all $x\in D$. Therefore, the inequality

$$\frac{\sum_{i=1}^{\alpha}\lambda_i f\left(x,y^i\right)}{\sum_{i=1}^{\alpha}\lambda_i g\left(x,y^i\right)}\geq\frac{\sum_{i=1}^{\alpha}\lambda_i f\left(u,y^i\right)+\sum_{j=1}^{k}\xi_j h_j\left(u\right)}{\sum_{i=1}^{\alpha}\lambda_i g\left(u,y^i\right)}$$

holds for all $x \in D$. Since $\lambda_i \geq 0$, $i = 1, \ldots, \alpha$, therefore, there exists i^* such that

$$\frac{f\left(x, y^{i^*}\right)}{g\left(x, y^{i^*}\right)} \geq \frac{\sum_{i=1}^{\alpha} \lambda_i f\left(u, y^i\right) + \sum_{j=1}^{k} \xi_j h_j\left(u\right)}{\sum_{i=1}^{\alpha} \lambda_i g\left(u, y^i\right)}.$$

Therefore,

$$\sup_{y\in Y} \frac{f(x, y)}{g(x, y)} \geq \frac{\sum_{i=1}^{\alpha} \lambda_i f\left(u, y^i\right) + \sum_{j=1}^{k} \xi_j h_j\left(u\right)}{\sum_{i=1}^{\alpha} \lambda_i g\left(u, y^i\right)}.$$

This means that the weak duality in the sense of Bector holds between (FP) and (FD4) and completes the proof of theorem. ■

Theorem 26 (*Strong Duality*). *Let $\bar{x}$ be an optimal point in* (FP) *and the Linear Independence Constraint Qualification (LICQ)* [25] *be satisfied at $\bar{x}$. Then, there exist $\left(\bar{\alpha}, \bar{\lambda}, \bar{y}\right) \in A$ and $(\bar{x}, \bar{\xi}) \in H_5\left(\bar{\alpha}, \bar{\lambda}, \bar{y}\right)$ such that $\left(\bar{\alpha}, \bar{\lambda}, \bar{y}, \bar{x}, \bar{\xi}\right)$ is optimal in* (FD4). *If also the hypotheses of Theorem 25 hold, then the corresponding optimal values of* (FP) *and* (FD4) *are equal.*

Proof. By assumption, $\bar{x}$ is an optimal point of (FP) and the Linear Independence Constraint Qualification (LICQ) holds at $\bar{x}$. Then, by (15)–(17), it follows that $\left(\bar{\alpha}, \bar{\lambda}, \bar{y}, \bar{x}, \bar{\xi}\right)$ is feasible for (FD4). Since

$$\frac{\sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i f\left(\bar{x}, \bar{y}^i\right) + \sum_{j=1}^{k} \bar{\xi}_j h\left(\bar{x}\right)}{\sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i g\left(\bar{x}, \bar{y}^i\right)} = \sup_{y\in Y} \frac{f(\bar{x}, y)}{g(\bar{x}, y)},$$

then, using the weak duality theorem (Theorem 25), we conclude that $\left(\bar{\alpha}, \bar{\lambda}, \bar{y}, \bar{x}, \bar{\xi}\right)$ is optimal for (FP). Therefore, the corresponding optimal values of (FP) and (FD4) are equal. ■

Theorem 27 (*Converse Duality*). *Let $\left(\bar{\alpha}, \bar{\lambda}, \bar{y}, \bar{u}, \bar{\xi}\right)$ be an optimal point in* (FD4) *such that $\bar{u} \in D$. Moreover, assume that Ψ is B-(p, r)-invex at $\bar{u}$ on $D \cup U_4$ with respect to η and b satisfying the condition: $b\left(x, \bar{u}\right) > 0$ for all $x \in D$. Then $\bar{u}$ is optimal in* (FP).

Proof. By assumption, Ψ is B-(p, r)-invex at $\bar{u}$ on $D \cup U_4$ with respect to η and b. Then, by Definition 1, we have

$$\frac{1}{r} b\left(x, u\right)\left(e^{r(\Psi(x)-\Psi(\bar{u}))} - 1\right) \geq \frac{1}{p} \nabla\Psi(\bar{u})\left(e^{p\eta(x,\bar{u})} - \mathbf{1}\right).$$

From the feasibility of $\left(\bar{\alpha}, \bar{\lambda}, \bar{y}, \bar{u}, \bar{\xi}\right)$ in (FD4), it follows that

$$\frac{1}{r} b\left(x, \bar{u}\right)\left(e^{r(\Psi(x)-\Psi(\bar{u}))} - 1\right) \geq 0.$$

Since $b\left(x, \bar{u}\right) > 0$ for all $x \in D$, then the inequality above gives

$$\Psi(x) \geq \Psi(\bar{u}).$$

By definition of Ψ, it follows that $\Psi(\bar{u}) = 0$. Thus,

$$\sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i g\left(\bar{u}, y^i\right)\left[\sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i f\left(x, y^i\right) + \sum_{j=1}^{k} \bar{\xi}_j h_j\left(x\right)\right] - \sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i g\left(x, y^i\right)\left[\sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i f\left(\bar{u}, y^i\right) + \sum_{j=1}^{k} \bar{\xi}_j h_j\left(\bar{u}\right)\right] \geq 0.$$

From the feasibility of x in (FD), we have $\sum_{j=1}^{k} \xi_j h_j\left(x\right) \leq 0$. Hence, the inequality

$$\sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i g\left(\bar{u}, y^i\right) \sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i f\left(x, y^i\right) \geq \sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i g\left(x, y^i\right)\left[\sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i f\left(\bar{u}, y^i\right) + \sum_{j=1}^{k} \bar{\xi}_j h_j\left(\bar{u}\right)\right]$$

holds for all $x \in D$. By assumption, $\bar{u} \in D$. Hence, by (64), $\sum_{j=1}^{k} \bar{\xi}_j h_j\left(\bar{u}\right) \leq 0$. Thus,

$$\sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i g\left(\bar{u}, y^i\right) \sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i f\left(x, y^i\right) \geq \sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i g\left(x, y^i\right) \sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i f\left(\bar{u}, y^i\right).$$

Therefore, the following inequality

$$\frac{\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i f\left(x, y^i\right)}{\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i g\left(x, y^i\right)} \geq \frac{\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i f\left(\overline{u}, y^i\right)}{\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i g\left(\overline{u}, y^i\right)} \tag{67}$$

holds for all $x \in D$. Suppose, contrary to the result, i.e. let there exists $\widetilde{x} \in D$ such that

$$\sup_{y \in Y} \frac{f(\widetilde{x}, y)}{g(\widetilde{x}, y)} < \sup_{y \in Y} \frac{f(\overline{u}, y)}{g(\overline{u}, y)}.$$

Since $\overline{\lambda}_i \geq 0$, $i = 1, \ldots, \overline{\alpha}$, then the inequality

$$\frac{\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i f\left(\widetilde{x}, y^i\right)}{\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i g\left(\widetilde{x}, y^i\right)} < \frac{\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i f\left(\overline{u}, y^i\right)}{\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i g\left(\overline{u}, y^i\right)}$$

contradicts (67). This completes the proof. ■

Theorem 28 (*Strictly Converse Duality*). *Let $\overline{x}$ and $\left(\overline{\alpha}, \overline{\lambda}, \overline{y}, \overline{u}, \overline{\xi}\right)$ be optimal points in* (FP) *and* (FD4) *and the Linear Independence Constraint Qualification (LICQ)* [25] *be satisfied at $\overline{x}$. Further, assume that Ψ is strictly B-(p, r)-invex at $\overline{u}$ on $D \cup U_4$ with respect to η and b. Then $\overline{x} = \overline{u}$, that is, $\overline{u}$ is optimal in* (FP).

Proof. Proof of this theorem is similar to the proof of Theorem 25 and, therefore, it was omitted in the paper. ■

We consider the following dual problem also in the sense of Bector

$$\max_{(\alpha,\lambda,\overline{y}) \in A} \quad \sup_{(u,\xi) \in H_5(\alpha,\lambda,\overline{y})} \frac{\sum_{i=1}^{\alpha} \lambda_i f\left(u, y^i\right)}{\sum_{i=1}^{\alpha} \lambda_i g\left(u, y^i\right)},$$

where $H_5\left(\alpha, \lambda, y\right)$ as the set of all $(u, \xi) \in X \times R_+^k$ satisfying the following conditions:

$$\nabla \left[\frac{\sum_{i=1}^{\alpha} \lambda_i f\left(u, y^i\right)}{\sum_{i=1}^{\alpha} \lambda_i g\left(u, y^i\right)} + \sum_{j=1}^{k} \xi_j h_j(u) \right] = 0, \tag{68}$$

(FD5)

$$\sum_{j=1}^{m} \xi_j h_j(u) \geq 0, \tag{69}$$

$$\lambda_i \geq 0, \sum_{i=1}^{\alpha} \lambda_i = 1, y^i \in Y\left(u\right), i = 1, \ldots, \alpha, \xi_j \geq 0, j = 1, \ldots, k. \tag{70}$$

The above dual problem, we also define in the following form

$$\max_{(\alpha,\lambda,\overline{y}) \in A} \quad \sup_{(u,\xi) \in H_5(\alpha,\lambda,\overline{y})} \frac{\sum_{i=1}^{\alpha} \lambda_i f\left(u, y^i\right)}{\sum_{i=1}^{\alpha} \lambda_i g\left(u, y^i\right)},$$

subject to (69)– (70) and (71) in place of (68) (FD5p)

$$\begin{aligned} &\frac{1}{p} \nabla \left[\frac{\sum_{i=1}^{\alpha} \lambda_i f\left(u, y^i\right)}{\sum_{i=1}^{\alpha} \lambda_i g\left(u, y^i\right)} + \sum_{j=1}^{k} \xi_j h_j(u) \right] \left(e^{p\eta(x,u)} - \mathbf{1}\right) \geq 0 \quad \text{if } p \neq 0 \\ &\nabla \left[\frac{\sum_{i=1}^{\alpha} \lambda_i f\left(u, y^i\right)}{\sum_{i=1}^{\alpha} \lambda_i g\left(u, y^i\right)} + \sum_{j=1}^{k} \xi_j h_j(u) \right] \eta(x, u) \geq 0 \quad \text{if } p = 0 \end{aligned} \quad \text{for all } x \in D. \tag{71}$$

Also in this case of Bector type duality, by (69)–(71), we define a class of Bector dual problems with respect to the function η and the scalar p.

If for a triplet $(\alpha, \lambda, \overline{y}) \in A$ the set $H_5\,(\alpha, \lambda, \overline{y})$ is empty, then we define the supremum over it to be $-\infty$.

By W_{FD5} we denote (assumed to be nonempty) the set of all feasible solutions of (FDP5). Moreover, let U_5 denote

$$U_5 = \{u \in X : (\alpha, \lambda, \overline{y}, u, \xi) \in W_{FD5}\}.$$

Comparing (FD3) and (FD5), we observe that (FD5) is some modification of (FD3). Hence, the statements and proofs of all the duality theorems for (FP)–(FD3) and (FP)–(FD5) are almost identical. However, we prove all duality theorems for (FP)–(FD5) under different assumption imposed on the functions constituting these problems than in the case of duality theorems for (FP)–(FD3).

Theorem 29 (*Weak Duality*). *Let x and $(\alpha, \lambda, y, u, \xi)$ be feasible solutions for* (FP) *and* (FD5), *respectively. Moreover, assume that the function $\frac{\sum_{i=1}^{\alpha}\lambda_i f_i(\cdot, y^i)}{\sum_{i=1}^{k}\lambda_i g_i(\cdot, y^i)} + \sum_{j=1}^{k}\xi_j h_j(\cdot)$ is B-(p, r)-invex at u on $D \cup U_5$ with respect to η and b satisfying $b\,(x, u) > 0$. Then the weak duality holds between* (FP) *and* (FD5).

Proof. By means of contradiction, we suppose that

$$\sup_{y\in Y}\frac{f(x,y)}{g(x,y)} < \frac{\sum_{i=1}^{\alpha}\lambda_i f\,(u, y^i)}{\sum_{i=1}^{\alpha}\lambda_i g\,(u, y^i)}.$$

Then by (70), it follows that

$$\frac{\sum_{i=1}^{\alpha}\lambda_i f(x, y^i)}{\sum_{i=1}^{\alpha}\lambda_i g(x, y^i)} < \frac{\sum_{i=1}^{\alpha}\lambda_i f\,(u, y^i)}{\sum_{i=1}^{\alpha}\lambda_i g\,(u, y^i)}.$$

Thus, from the feasibility of x and $(\alpha, \lambda, y, u, \xi)$ in (FP) and (FD5), respectively, we get

$$\frac{\sum_{i=1}^{\alpha}\lambda_i f_i(x, y^i)}{\sum_{i=1}^{\alpha}\lambda_i g_i(x, y^i)} + \sum_{j=1}^{m}\xi_j h_j(x) < \frac{\sum_{i=1}^{\alpha}\lambda_i f_i(u, y^i)}{\sum_{i=1}^{\alpha}\lambda_i g_i(u, y^i)} + \sum_{j=1}^{m}\xi_j h_j(u). \tag{72}$$

By assumption, the function $\frac{\sum_{i=1}^{\alpha}\lambda_i f_i(\cdot, y^i)}{\sum_{i=1}^{\alpha}\lambda_i g_i(\cdot, y^i)} + \sum_{j=1}^{k}\xi_j h_j(\cdot)$ is B-(p, r)-invex at u on $D \cup U_5$ with respect to η and b. Thus, by Definition 1, it follows that

$$\frac{1}{r}b\,(x,u)\left[e^{r\left(\frac{\sum_{i=1}^{\alpha}\lambda_i f_i(x,y^i)}{\sum_{i=1}^{\alpha}\lambda_i g_i(x,y^i)}+\sum_{j=1}^{m}\xi_j h_j(x)\right)-r\left(\frac{\sum_{i=1}^{\alpha}\lambda_i f_i(u,y^i)}{\sum_{i=1}^{\alpha}\lambda_i g_i(u,y^i)}+\sum_{j=1}^{m}\xi_j h_j(u)\right)} - 1\right] \geq \nabla\left[\frac{\sum_{i=1}^{\alpha}\lambda_i f\,(u, y^i)}{\sum_{i=1}^{\alpha}\lambda_i g\,(u, y^i)} + \sum_{j=1}^{k}\xi_j h_j(u)\right]\left(e^{p\eta(x,u)} - \mathbf{1}\right). \tag{73}$$

From the hypothesis, $b\,(x, u) > 0$. Then, by (72) and (73), we get the inequality

$$\nabla\left[\frac{\sum_{i=1}^{\alpha}\lambda_i f\,(u, y^i)}{\sum_{i=1}^{\alpha}\lambda_i g\,(u, y^i)} + \sum_{j=1}^{k}\xi_j h_j(u)\right]\left(e^{p\eta(x,u)} - \mathbf{1}\right) < 0,$$

which contradicts the feasibility of $(\alpha, \lambda, y, u, \xi)$ in (FD5). ■

Theorem 30 (*Strong Duality*). *Let $\overline{x}$ be an optimal point for* (FP) *and the Linear Independence Constraint Qualification (LICQ)* [25] *be satisfied at $\overline{x}$. Then, there exist $\left(\overline{\alpha}, \overline{\lambda}, \overline{y}\right) \in A$ and $(\overline{x}, \overline{\xi}) \in H_5\left(\overline{\alpha}, \overline{\lambda}, \overline{y}\right)$ such that $\left(\overline{\alpha}, \overline{\lambda}, \overline{y}, \overline{x}, \overline{\xi}\right)$ is optimal in* (FD5) *and the corresponding objective values of* (FP) *and* (FD5) *are equal. If also the hypotheses of* Theorem 29 *are satisfied, then $\left(\overline{\alpha}, \overline{\lambda}, \overline{y}, \overline{x}, \overline{\xi}\right)$ is optimal in* (FD5).

Proof. Proof is the same as for the strong duality theorem for Bector's dual problem (FD3) (Theorem 22). ■

Theorem 31 (*Converse Duality*). *Let $\left(\overline{\alpha}, \overline{\lambda}, \overline{y}, \overline{u}, \overline{\xi}\right)$ be an optimal solution in* (FP) *such that $\overline{u} \in D$. Further, we assume that the function $\frac{\sum_{i=1}^{\overline{\alpha}}\overline{\lambda}_i f_i(\cdot, \overline{y}^i)}{\sum_{i=1}^{\overline{\alpha}}\overline{\lambda}_i g_i(\cdot, \overline{y}^i)} + \sum_{j=1}^{k}\overline{\xi}_j h_j(\cdot)$ is B-(p, r)-invex at $\overline{u}$ on $D \cup U_5$ with respect to η and b satisfying $b\,(x, \overline{u}) > 0$ for all $x \in D$. Then $\overline{u}$ is optimal in* (FP).

Proof. Proof of theorem is similar to the proof of Theorem 27. ■

Theorem 32 (*Strictly Converse Duality*). *Let* $\bar{x}$ *and* $\left(\bar{\alpha}, \bar{\lambda}, \bar{y}, \bar{u}, \bar{\xi}\right)$ *be optimal solutions in* (FP) *and* (FD5). *Further, assume that the Linear Independence Constraint Qualification (LICQ)* [25] *is satisfied at* $\bar{u}$*, and, moreover, the function* $\frac{\sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i f_i(\cdot, \bar{y}^i)}{\sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i g_i(\cdot, \bar{y}^i)} + \sum_{j=1}^{k} \bar{\xi}_j h_j(\cdot)$ *is strictly* B-(p, r)*-invex at* $\bar{u}$ *on* $D \cup U_5$ *with respect to* η *and* b. *Then* $\bar{x} = \bar{u}$, *that is,* $\bar{u}$ *is an optimal point for* (FP).

Proof. Suppose, contrary to the result, that $\bar{x} \neq \bar{u}$, and hence $\bar{u}$ is not optimal in (FP). Then,

$$\sup_{y \in Y} \frac{f(\bar{x}, y)}{g(\bar{x}, y)} < \sup_{y \in Y} \frac{f(\bar{u}, y)}{g(\bar{u}, y)}.$$

From the feasibility of $\tilde{x}$ in (FP) together with (69) and (70), we obtain

$$\frac{\sum_{i=1}^{\alpha} \bar{\lambda}_i f_i(\bar{x}, \bar{y}^i)}{\sum_{i=1}^{\alpha} \bar{\lambda}_i g_i(\bar{x}, \bar{y}^i)} + \sum_{j=1}^{m} \bar{\xi}_j h_j(\bar{x}) < \frac{\sum_{i=1}^{\alpha} \bar{\lambda}_i f_i(\bar{u}, \bar{y}^i)}{\sum_{i=1}^{\alpha} \bar{\lambda}_i g_i(\bar{u}, \bar{y}^i)} + \sum_{j=1}^{m} \bar{\xi}_j h_j(\bar{u}). \tag{74}$$

By assumption, $\frac{\sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i f_i(\cdot, \bar{y}^i)}{\sum_{i=1}^{\bar{\alpha}} \bar{\lambda}_i g_i(\cdot, \bar{y}^i)} + \sum_{j=1}^{k} \bar{\xi}_j h_j(\cdot)$ is strictly B-(p, r)-invex at $\bar{u}$ on $D \cup U_5$ with respect to η and with respect to b. Then, by Definition 1, there exists a function b such that $b(\bar{x}, \bar{u}) > 0$. Hence, by (74),

$$\frac{1}{r} b(\bar{x}, \bar{u}) \left(e^{r\left(\frac{\sum_{i=1}^{\alpha} \bar{\lambda}_i f_i(\bar{x}, \bar{y}^i)}{\sum_{i=1}^{\alpha} \bar{\lambda}_i g_i(\bar{x}, \bar{y}^i)} + \sum_{j=1}^{m} \bar{\xi}_j h_j(\bar{x}) - \frac{\sum_{i=1}^{\alpha} \bar{\lambda}_i f_i(\bar{u}, \bar{y}^i)}{\sum_{i=1}^{\alpha} \bar{\lambda}_i g_i(\bar{u}, \bar{y}^i)} + \sum_{j=1}^{m} \bar{\xi}_j h_j(\bar{u}) \right)} - 1 \right) < 0.$$

Then, by Definition 1, we get the following inequality

$$\frac{1}{p} \nabla \left[\frac{\sum_{i=1}^{\alpha} \bar{\lambda}_i f_i(\bar{u}, \bar{y}^i)}{\sum_{i=1}^{\alpha} \bar{\lambda}_i g_i(\bar{u}, \bar{y}^i)} + \sum_{j=1}^{m} \bar{\xi}_j h_j(\bar{u}) \right] \left(e^{p\eta(\bar{x}, \bar{u})} - \mathbf{1} \right) < 0,$$

which contradicts the duality constraint (68). This completes the proof. ■

5. Conclusion

In this paper, we have established some sufficient optimality conditions and several parametric and parameter-free duality results for a class of smooth generalized fractional minimax programming problems possessing some B-(p, r)-invexity property. This paper extends entirely earlier works, in which optimality conditions and duality results have been obtained for a generalized fractional optimization problem by applying a convexity assumption or a generalized convexity assumption imposed on functions constituting it (see, for example, [12,3,10,27]). Also optimality conditions and some duality results, under invexity assumption for optimization problems of this type, have been extended (see, for example, [14]).

In the case when Y is a singleton, the considered generalized fractional programming problems (FP) becomes the standard fractional problem and duals reduce to the well known duals of Schaible [27], Bector [12] and Weir [2], respectively.

Evidently, all the optimality and duality results established in this paper for the considered generalized fractional programming problem (FP) generalize earlier results for this type of optimization problems (for example, the results of Chandra and Kumar [24], Zalmai [14] and Antczak [17]). In the paper, we have presented not single models of the specified type, but we have defined various classes of dual models of the specified type with respect to the function η and the scalar p. In particular, the duality models of Section 4 have not been considered previously for such a wide class of generalized minimax programming problems, that is, involving B-(p, r)-invex functions.

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